18.03 Class 27, April 12, 2010

Laplace Transform II

1. Delta signal
2. t-derivative rule
3. Inverse transform
4. Unit impulse response
5. Partial fractions
6. L[f'_r]

Laplace Transform: F(s) = int_0^\infty f(t) e^{-st} dt

We saw that this improper integral may only converge for Re(s) > a, for some a depending upon f(t) . The smallest such a gives the "region of convergence."

Computations so far:

L[1] = 1/s

L[t^n] = n!/s^{n+1}

L[e^{rt}] = 1/(s-r)

L[cos(t)] = s / (s^2 + omega^2)

L[sin(t)] = omega / (s^2 + omega^2)

We also have

Rule 1 (Linearity): af(t) + bg(t) -----> aF(s) + bG(s).

Rule 2 (s-shift):   L[e^{rt}f(t)] = F(s-r)

Warm-up: What is the Laplace transform of f(t) = e^{-t} cos(3t) ?

We could do this by writing it as (1/2)(e^{(-1+3i)t} + e^{(-1-3i)t})

but it's a bit easier to use the s-shift : cos(3t) ----> s / (s^2 + 9)

so   e^{-t} cos(3t) ----> (s+1)/((s+1)^2 + 9)

If you like you can "uncomplete the square" and write this as

= (s+1) / (s^2 + 2s + 10)

[1] The delta function: For b > 0 ,

L[delta(t-b)] = integral_0^infty delta(t-b) e^{-st} dt

What could this mean?

If f(t) is continuous at t = b , delta(t-b) f(t) = f(b) delta(t-b) .

so integral delta(t-b) f(t) dt = f(b) u(t-b) + const

or, if a < b < c , integral_a^c delta(t-b) f(t) dt = f(b) .

In some accounts, this is the DEFINITION of the delta function.

In our situation, you get

L[delta(t-b)] = e^{-bs}

In this case, the region of convergence is the entire complex plane: the limit you take to get the improper integral is constant as soon as t > b.

There is a problem when b = 0 . delta(t) e^{-st} = delta(t) for any s , but int_0^infty delta(t) dt = u(infty) - u(0) and u(0) is indeterminate.

We want the formula that worked for b > 0 to work for b = 0 as well:

L[delta(t)] = 1

To be sure this happens, we should refine the definition of the LT integral so the lower limit (as well as the upper limit) occurs as a limit:

Refinement #2:

L[f(t)] = lim_{c increasing to infty, a increasing to 0}

int_a^c f(t) e^{-st} dt

and then we have the new computation

L[delta(t-b)] = e^{-bs} for b greater or equal to 0
region of convergence: the whole plane.

[2] To use LT in understanding differential equations, we will need:

L[f'(t)] = integral_{0-}^infty f'(t) e^{-st} dt

Parts: u = e^{-st} du = -s e^{-st} dt

dv = f'(t) dt v = f(t)

... = e^{-st} f(t) |_{0-}^infty + s integral f(t) e^{-st} dt

= s F(s)

Now, what is f'(t) ? If f(t) has discontinuities, we must mean the generalized derivative; that's the only way to make the integral of the derivative work right. Even if f(t) has no breaks in its graph for t > 0 , it probably will have one when t = 0 since we are assuming that f(0-) = 0 but have not assumed that f(0+) = 0 . We have to expect a discontinuity at t = 0 , and so a delta function in f'(t) at t = 0 .

For example, $u(t) = f(t) \longrightarrow 1/s$
and $\delta(t) = f'(t) \longrightarrow s\,(1/s) = 1$

or $t^n = f(t) \longrightarrow n!/s^{n+1}$
and $n\,t^{n-1} = f'(t) \longrightarrow s\,n!/s^{n+1} = n\,(n-1)!/s^n$

or $\cos(t) = f(t) \longrightarrow s/(s^2+1)$
$\delta(t) - \sin(t) = f'(t) \longrightarrow s^2/(s^2+1) = 1 - 1/(s^2+1)$

[3] In summary the use of Laplace transform in solving initial value problems goes like this:

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                         L
      IVP for  x(t)   --------->  Alg equation for  X(s)
                                           |
                                           | solve
                                           |
                      L^{-1}               V
      x(t) = ...      <---------      X(s) = ...
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For this to work we have to recover information about $f(t)$ from $F(s)$. There isn't a formula for L^{-1}; what one does is look for parts of $F(s)$ in our table of computations. It's an art, like integration. There is no free lunch.

We can't expect to recover $f(t)$ exactly, if $f(t)$ isn't required to be continuous, since $F(s)$ is defined by an integral, which is left unchanged if we alter any individual value of $f(t)$. What we have is:

Theorem: If $f(t)$ and $g(t)$ are generalized functions with the same Laplace transform, then for every a greater or equal to 0
$f(a+) = g(a+)$, $f(a-) = g(a-)$, and any
occurrences of delta functions are the same in $f(t)$ as in $g(t)$.

So if $f(t)$ and $g(t)$ are continuous at $t = a$, then $f(a) = g(a)$.

[4] Example: Find the unit impulse response for the operator $D + 3I$

ie solve $w' + 3w = \delta(t)$ with rest initial conditions.

Step 1: Apply L : $sW + 3W = 1$

Step 2: Solve for W : $W = 1/(s+3)$

Step 3: Find $w(t)$ with this as Laplace transform: e^{-3t}

or more precisely, $u(t)\,e^{-3t}$.

Laplace transform is a good way to find unit impulse responses.

"Unit impulse response" = "weight function"

Its Laplace transform is called the "transfer function."

[5] Example: Solve $x' + 3x = e^{-t}$, with initial condition $x(0+) = 0$.

Step 1: Apply L : $sX + 3X = 1/(s+1)$, using linearity, the s-shift rule, and the t-derivative rule.

Step 2: Solve for X: $(s+3)X = 1/(s+1)$

so $X = 1/((s+1)(s+3))$

Step 3: Massage the result into a linear combination of recognizable forms.

Here the technique is:

Partial Fractions: $1/((s+1)(s+3)) = a/(s+1) + b/(s+3)$.

Old method: cross multiply and identify coefficients.

This works fine, but for excitement let me offer:

The Cover-up Method: Step (i) Multiply through by $(s+1)$:

$$1/(s+3) = a + (s+1)(a/(s+3))$$

Step (ii) Set $s + 1 = 0$, or $s = -1$:

$$1/(3-1) = a + 0 : \quad a = 1/2 .$$

This process "covers up" occurrences of the factor $(s+1)$, and also all unwanted unknown coefficients. The same method gives b :

$$1/(-3+1) = 0 + b : \quad b = -1/2.$$

So $$X = (1/2)/(s+1) - (1/2)/(s+3)$$

Step 4: Apply L^{-1}: we can now recognize both terms:

$$x = (1/2) e^{-t} - (1/2) e^{-3t} . \qquad \text{(times } u(t))$$

Of course, this is very easy to do by our earlier methods: The ERF gives the first term, the general solution to the homogeneous equation is ce^{-3t} , and the transient needed for initial condition $x(0) = 0$ is $c = -1/2$.

[5] Consider the equation

$$x' + 3x = e^{-t} , \quad x(0) = 5$$

Since we have the standing agreement that $x(t) = 0$ for $t < 0$, $x(t)$ has a jump, apparently, at $t = 0$, and perhaps what is intended is

$$x' + 3x = e^{-t} , \quad x(0+) = 5$$

But this equation does not have a solution! Since $x(0-) = 0$, x' contains the singular part $5 \delta(t)$; but there's no $5 \delta(t)$ on the right hand side.

What is really intended in a problem like this is, in connection with LT is:

$$x'_r + 3x = e^{-t} , \quad x(0+) = 0 .$$

Just to keep the notation in bounds, let's suppose that $f(t)$ is continuous for $t > 0$. Then the only singular part of the generalized derivative occurs at $t = 0$:

$$(f')_s(t) = f(0+) \delta (t)$$

The generalized derivative is the sum of this and the ordinary derivative $(f')_r(t)$. By linearity and our value $L[\delta(t)] = 1$,

$$L[f'(t)] = f(0+) + L[f'_r(t)]$$

and so

$$L[f'_r(t)] = s F(s) - f(0+) .$$

This is what you always see in books and most of the time it's what is used in practice. Let's use it:

$$x'_r + 3x = e^{-t}$$

Then $(sX - 5) + 3X = 1/(s+1)$

$$X = 1 / (s+1)(s+3) + 5/(s+3)$$

so we add $L^{-1}[5/(s+3)] = 5 e^{-3t}$ to the earlier solution -- this corrects the transient.

Current list of Rules:

L is linear: $L(af+bg) = aF + bG$

s-shift: $e^{at}f(t) \longrightarrow F(s-a)$

t-derivative: $f'(t) \longrightarrow s F(s)$

$f'_r(t) \longrightarrow s F(s) - f(0+)$
if $f(t)$ has continuous derivative for $t > 0$.

Computations:

$1 \longrightarrow 1/s$

$e^{as} \longrightarrow 1/(s-a)$

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cos(omega t) ----> s/(s^2+omega^2)

sin(omega t) ----> omega/(s^2+omega^2)

delta(t-a)   ----> e^{as}
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