18.03 Problem Set 3: First Half

This is the first two problems of PS3. The rest will be available on February 26.
I encourage collaboration on homework in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

I. First-order differential equations

L8	F 19 Feb	Autonomous equations; the phase line, stability: EP 1.7, 7.1.
L9	M 22 Feb	Wrap-up: Linear vs Nonlinear.
R6	T 23 Feb	Exam preparation.
L10	W 24 Feb	**Hour Exam I**

II. Second-order linear equations

R7	Th 25 Feb	Harmonic oscillator, superposition
L11	F 26 Feb	The spring-mass-dashpot model; characteristic polynomial; Solution in real root case: EP 2.1, 2.2.

Part I.

8. (F 19 Feb) Notes: 1E-1.

9. (M 22 Feb) Consider the differential equation $\dot{x} + 2x = 1$.
(a) Find the general solution (i) by separation; (ii) by use of an integrating factor; (iii) by regarding the right hand side as e^{0t} and using the method of optimism (i.e. look for a solution of the form Ae^{0t}) to find a particular solution, and then adding in a transient.
(b) This equation is also autonomous. Sketch its phase line and some solutions (including the equilibrium solution). Is the equilibrium stable, unstable, or neither?
(c) Use Euler's method with three steps to estimate the value of the solution with initial condition $x(0) = 0$ at $t = 1$.

Part II.

8. (F 19 Feb) [Autonomous Equations] This problem will use the Mathlet `Phase Lines`. Open the applet and understand its use and conventions. Click on [Phase Line] to see a representation of the phase line. Note the color coding: a green dot represents a stable or attracting equilibrium; red represents an unstable or repelling equilibrium; and blue represents a "semi-stable" equilibrium.

The Kenyan government has a game preserve that, in the absence of hunting, supports an oryx population that follows the logistic equation with a stable population of one kilo-oryx (one thousand animals). Kenya wishes to investigate the effect on the oryx population of various rates hunting, a.

(a) This situation is well modeled by the top menu item in `Phase Lines`. Explain why this is a good model. The rest of this problem will use this equation.

(b) It appears that there is a pair of equilibria for some values of a, only one for at least one other value of a, and none for still other values of a. Calculate which values of a behave in which way; for each a find the critical values of y, and in each case say whether the critical points involved are stable, unstable, or semi-stable.

(c) The Kenyan government hopes to allow 187.5 oryx to be killed in an average year. Determine for them what the resulting stable population will be. If this strategy is adopted, what is the critical oryx population below which the population will crash (if the same harvest rate continues to be allowed)?

(d) For this value of a, there are five different behaviors possible for the oryx population. (Two solutions exhibit the "same behavior" if one is a time-translate of the other). Sketch one solution of each of the five types. Your sketch should make it clear what the behavior of the solution is as t gets small and as t gets large. Match each one up with a portion of the phase line.

(e) Invoke the Bifurcation Diagram for this autonomous equation. Move a along its slider to see the variety of behaviors of the phase line of as a varies. The green and red curve in the newly displayed bifurcation plane represents the equilibrium points for those equations, for various values of a. Give an equation for that curve.

9. (M 22 Feb) [Linear vs Nonlinear] Still working with the equation $\dot{y} = (1-y)y - a$ with $a = \frac{3}{16}$, let y_0 be the stable critical point. Write $u = y - y_0$ for the population excess over equilibrium (so $u < 0$ if the population is less than the equilibrium value).

(a) Rewrite the differential equation as a differential equation for u. Check that the new equation is again autonomous and that $u = 0$ is a critical point for it.

(b) For small u we can neglect higher powers of u (such as u^2). This process is "linearization near equilibrium." What is the linearized equation near $u = 0$? What is the general solution of this linear autonomous equation?

(c) At least in this case, when solutions of the original autonomous equation get near equilibrium, they are well modeled by solutions of the linearization. Give an approximation of y near equilibrium. Use it to answer this question: if $y(10) - y_0 = b$, estimate $y(11)$ and $y(12)$.

(d) Suppose that the linear equation $\dot{x} + p(t)x = q(t)$ is autonomous. What can you say about $p(t)$ and $q(t)$?

Part I solutions. 9(a)(iii) $e^{0t} = 1$ so we are seeking a constant solution $x = A$: $\dot{x} + 2x = 2A$ so $A = \frac{1}{2}$.

9(c) $\frac{13}{27}$.

18.03 Differential Equations
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