Recitation 5, February 18, 2010

Complex numbers, complex exponentials

Solution suggestions

1. Mark $z = 1 + \sqrt{3}i$ on the complex plane. What is its polar coordinates? Then mark z^n for $n = 1, 2, 3, 4$. What is each in the form $a + bi$? What is each one in the form $Ae^{i\theta}$? Then mark z^n for $n = 0, -1, -2, -3, -4$.

For z, $r = 2$ and $\theta = \pi/3$, so it's on the radial line of $\pi/3$, with a distance 2 from the origin. z^2 has argument $2\pi/3$ and radius 4, so by Euler's formula, $z^2 = 4e^{i2\pi/3} = -2 + 2\sqrt{3}i$. z^3 has argument π and radius 8, so it's equal to -8. z^4 has argument $4\pi/3$ and radius 16, so it's equal to $-8 - 8\sqrt{3}i$. z^n has the form $2^n e^{in\pi/3}$, for $n = 1, 2, 3, 4$. $z^0 = 1$. The negative powers lie in the bottom half of the complex plane. z^{-k} is on the radial line of $-k\pi/3$ with radius 2^{-k} for $k = 1, 2, 3, 4$.

2. Find a complex number $a+bi$ such that $e^{a+bi} = 1+\sqrt{3}i$. In fact, find all such complex numbers. For definiteness, fix b to be positive but as small as possible. (This is probably the first one you thought of.) What is $e^{n(a+bi)}$ for $n = 1, 2, 3, 4$? (Hint: $e^{n(a+bi)} = (e^{a+bi})^n$.) How about for $n = 0, -1, -2, -3, -4$?

$1 + \sqrt{3}i$ has modulus 2 and argument $\pi/3 + 2k\pi$ for all integers k, so $1 + \sqrt{3}i$ can be expressed as a complex exponential of the form $2e^{i(\pi/3+2k\pi)}$. Taking logs gives us the equation $a + bi = \ln 2 + i(\pi/3 + 2k\pi)$. The smallest positive value of b is $\pi/3$. Following the hint, $e^{n(a+bi)} = (1 + \sqrt{3}i)^n$, so by question 1, for $n = 1, 2, 3, 4$, the answer is $1+\sqrt{3}i, -2+2\sqrt{3}i, -8, -8-8\sqrt{3}i$, respectively. For $n = 0, -1, -2, -3, -4$, we have 1, $2^{-1}e^{-i\pi/3} = \frac{1-\sqrt{3}i}{4}$, $2^{-2}e^{-i2\pi/3} = \frac{-1-\sqrt{3}i}{8}$, $2^{-3}e^{-i\pi} = -1/8$, and $2^{-4}e^{-i4\pi/3} = \frac{-1+\sqrt{3}i}{32}$.

3. Write each of the following functions $f(t)$ in the form $A\cos(\omega t - \phi)$. In each case, begin by drawing a right triangle with sides a and b. (a) $\cos(2t) + \sin(2t)$. (b) $\cos(\pi t) - \sqrt{3}\sin(\pi t)$. (c) $\mathrm{Re}\dfrac{e^{it}}{2+2i}$.

(a). Here, our right triangle has hypotenuse $\sqrt{2}$, so $A = \sqrt{2}$. Both summands have "circular frequency" 2, so $\omega = 2$. ϕ is the argument of the hypotenuse, which is $\pi/4$, so $f(t) = \sqrt{2}\cos(2t - \pi/4)$.

(b). The right triangle has hypotenuse of length $\sqrt{1^2 + (-\sqrt{3})^2} = 2$. The circular frequency of both summands is π, so $\omega = \pi$. The argument of the hypotenuse is $-\pi/3$, so $f(t) = 2\cos(\pi t + \pi/3)$.

(c). $e^{it} = \cos(t) + i\sin(t)$, and $\frac{1}{2+2i} = \frac{1-i}{4}$. the real part is then $\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t)$. The right triangle here has hypotenuse $\frac{\sqrt{2}}{4}$ and argument $\pi/4$, so $f(t) = \frac{\sqrt{2}}{4}\cos(t - \pi/4)$.

4. Find a solution of $\dot{x}+2x = e^t$ of the form we^t. Do the same for $\dot{z}+2z = e^{2it}$. In both cases, go on to write down the general solution.

We can use integrating factors to get $(ux)' = ue^t$ for $u = e^{2t}$. Integrating yields $e^{2t}x = e^{3t}/3 + c$, or $x = e^t/3 + ce^{-2t}$. So when $c = 0$, $x = e^t/3$ is the solution of the required form. We can use the same integrating factors for z, since the equations have identical homogeneous parts. This gives us $e^{2t}z = e^{(2+2i)t}/(2+2i) + C$, or $z = e^{2it}/(2+2i) + Ce^{-2t}$, where C is any complex number. In particular, when $C = 0$, $z = e^{2it}/(2+2i)$.

5. Find a solution of $\dot{x} + 2x = \cos(2t)$ by replacing it with a complex valued equation, solving that, and then extracting the real part. Your work also gives you a solution for $\dot{x} + 2x = \sin(2t)$.

$\cos(2t) = \mathrm{Re}(e^{2it})$, so x can be the real part of any solution z to $\dot{z}+2z = e^{2it}$. In particular, from question 4, one solution is given by $x = \mathrm{Re}\left(e^{2it}/(2+2i)\right) = \frac{\cos(2t)+\sin(2t)}{4}$. The solution for $\dot{x} + 2x = \sin(2t)$ will be the corresponding imaginary part, i.e., $x = \frac{-\cos(2t)+\sin(2t)}{4}$. Note that the general solutions to $\dot{x} + 2x = \cos(2t)$ are given by $\frac{\cos(2t)+\sin(2t)}{4} + ce^{-2t}$.