LT. Laplace Transform

1. Translation formula. The usual L.T. formula for translation on the t-axis is

$$(1) \qquad \mathcal{L}\big(u(t-a)f(t-a)\big) \;=\; e^{-as}F(s), \qquad \text{where} \;\; F(s) = \mathcal{L}\big(f(t)\big)\;, \quad a > 0.$$

This formula is useful for computing the inverse Laplace transform of $e^{-as}F(s)$, for example. On the other hand, as written above it is not immediately applicable to computing the L.T. of functions having the form $u(t-a)f(t)$. For this you should use instead this form of (1):

$$(2) \qquad \mathcal{L}\big(u(t-a)f(t)\big) \;=\; e^{-as}\mathcal{L}\big(f(t+a)\big), \quad a > 0.$$

Example 1. Calculate the Laplace transform of $u(t-1)(t^2+2t)$.

Solution. Here $f(t) = t^2 + 2t$, so (check this!) $f(t+1) = t^2 + 4t + 3$. So by (2),

$$\mathcal{L}\big(u(t-1)(t^2+2t)\big) \;=\; e^{-s}\mathcal{L}(t^2+4t+3) \;=\; e^{-s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right) .$$

Example 2. Find $\mathcal{L}\big(u(t-\frac{\pi}{2})\sin t\big)$.

Solution.

$$\begin{aligned} \mathcal{L}\big(u(t-\tfrac{\pi}{2})\sin t\big) &= e^{-\pi s/2}\mathcal{L}\big(\sin(t+\tfrac{\pi}{2})\big) \\ &= e^{-\pi s/2}\mathcal{L}(\cos t) \;=\; e^{-\pi s/2}\frac{s}{s^2+1} . \end{aligned}$$

Proof of formula (2). According to (1), for any $g(t)$ we have

$$\mathcal{L}\big(u(t-a)g(t-a)\big) \;=\; e^{-as}\mathcal{L}\big(g(t)\big);$$

this says that to get the factor on the right side involving g, we should replace $t-a$ by t in the function $g(t-a)$ on the left, and then take its Laplace transform.

Apply this procedure to the function $f(t)$, written in the form $f(t) = f((t-a)+a)$; we get ("replacing $t-a$ by t and then taking the Laplace Transform")

$$\mathcal{L}\big(u(t-a)f((t-a)+a)\big) \;=\; e^{-as}\mathcal{L}\big(f(t+a)\big),$$

exactly the formula (2) that we wanted to prove. $\square$

Exercises. Find: a) $\mathcal{L}\big(u(t-a)e^t\big)$ b) $\mathcal{L}\big(u(t-\pi)\cos t\big)$ c) $\mathcal{L}\big(u(t-2)te^{-t}\big)$

Solutions. a) $e^{-as}\dfrac{e^a}{s-1}$ b) $-e^{-\pi s}\dfrac{s}{s^2+1}$ c) $e^{-2s}\dfrac{e^{-2}(2s+3)}{(s+1)^2}$

2. The transfer function and Green's function.

If we use the Laplace transform to solve the IVP

$$y'' + ay' + by \;=\; r(t), \qquad y(0) = 0,\; y'(0) = 0,$$

the transform of the IVP, with the usual notation, is

$$s^2 Y + asY + bY \;=\; R(s) \;;$$

whose solution for $Y = \mathcal{L}^{-1}(y)$ is

$$Y \;=\; R(s)\frac{1}{s^2 + as + b} \;;$$

using the convolution operator to take the inverse transform, we get as the solution (further down the function $w(t)$ is defined):

$$y \;=\; r(t) * w(t) \;=\; \int_0^t r(u)w(t-u)\,du \;. \tag{3}$$

In this form of the solution, the following terminology is often used. Let $p(D) = D^2 + aD + b$ be the differential operator; then we write

$$\begin{aligned} W(s) &= \frac{1}{s^2 + as + b} && \text{the } \textbf{transfer function} \text{ for } p(D), \\ w(t) &= \mathcal{L}^{-1}\big(W(s)\big) && \text{the } \textbf{weight function} \text{ for } p(D), \\ G(t,u) &= w(t-u) && \text{the } \textbf{Green's function} \text{ for } p(D). \end{aligned}$$

The important thing to note is that each of these depends only on the operator, not on the forcing function $r(t)$; once they are calculated, the solution (3) to the IVP can be written down immediately as the definite integral there, and used for a variety of different $r(t)$.

The weight function $w(t)$ can be thought of as the unique solution to the IVP

$$y'' + ay' + by \;=\; 0; \qquad y(0) = 0, \quad y'(0) = 1; \tag{4}$$

or as the solution to the IVP

$$y'' + ay' + by \;=\; \delta(t); \quad y(0) = 0, \quad y'(0^-) = 0; \tag{5}$$

in the second equation, $\delta(t)$ is the Dirac delta function. Both IVP's model (for $a, b > 0$) a damped spring-mass system which is initially at rest, but whose mass is given a unit impulse at time zero, say by a sharp blow.

It is an easy exercise to show that $w(t)$ is the solution to both IVP's. As an example of Green's functions, see the last few Laplace Transform exercises (in Section 3D).

18.03 Differential Equations
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