18.034 Honors Differential Equations
Spring 2009

18.034 Problem Set #2

Due by Friday, February 20, 2009, by NOON.

Notation. $' = d/dx$.

1. Let

$$f(x) = \begin{cases} x/|x| & \text{for} \quad x \neq 0, \\ k & \text{for} \quad x = 0, \end{cases}$$

where k is a constant. Show that no matter how the constant k is chosen, the differential equation $y' = f(x)$ has no solution on an interval containing the origin.

2. Suppose that f be a continuous bounded function for the entire real axis. If f' is continuous, then show that the nonzero solution of the initial value problem of $y' = yf(y)$ with $y(0) = y_0 \neq 0$ exists for all x. (You may need to assume the uniqueness theorem.)

3. Brikhoff-Rota, pp. 20, #9.

4. (The *Ricatti* equation) It is the differential equation of the form $y' = a(x) + b(x)y + c(x)y^2$. In general the Ricatti equation is not solvable by elementary means*. However,
(a) show that if $y_1(x)$ is a solution then the general solution is $y = y_1 + u$, where u is the general solution of a certain Bernoulli equation (cf. pset #1).

(b) Solve the Ricatti equation $y' = 1 - x^2 + y^2$ by the above method.

5. Let

$$Ly = y'' + y.$$

We are going to find the *rest solution* of the differential equation $Ly = 3\sin 2x + 3 + 4e^x$. That is the solution with $u(0) = u'(0) = 0$.
(a) Find the general solution of $Ly = 0$.

(b) Solve $Ly = 3\sin 2x$, $Ly = 3$, and $Ly = 4e^x$ by use of appropriate trial solutions.

(c) Determine the constants in

$$y(x) = c_1 \cos x + c_2 \sin x - \sin 2x + 3 + 2e^x$$

to find the solution.

6. (Euler's equi-dimensional equation) It is a differential equation of the form $x^2y'' + pxy' + qy = 0$, where p, q are constants.
(a) Show that the setting $x = e^t$ changes the differential equation into an equation with constant coefficients.

(b) Use this to find the general solution to $x^2y'' + xy' + y = 0$.

(c) For which values of p, the general solutions of $x^2y'' + pxy' + 2y = 0$ are defined for the entire real axis $(-\infty, \infty)$?

*This was shown by Liouville in 1841.