18.034 Honors Differential Equations
Spring 2009

18.034 Solutions to Problemset 4

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1. (a) $u' = u_1'v + u_1v'$, $u'' = u_1''v + 2u_1'v' + u_1v''$,
   $u''' = u_1'''v + 3u_1''v' + 3u_1'v'' + u_1v'''$.

   (b) The equation for v' reduces to $(2-x)v''' + (3-x)v'' = 0$, so that $v = -c_1xe^{-x} + c_2x + c_3$. Hence, $u = c_1x + c_2xe^x + c_3e^x$.

2. (a) $\{x^\alpha, x^\beta\}$ is a basis of solution of the homogeneous equation.

$$u(x) = \frac{1}{\alpha - \beta}\left(x^\alpha \int f(x)x^{1-\alpha}\,dx - x^\beta \int f(x)x^{1-\beta}\,dx\right)$$

   (b) $\{x^\alpha, x^\alpha \log x\}$ is a basis of the homogenous equation.

$$u(x) = x^\alpha \log x \int f(x)x^{1-\alpha}\,dx - x^\alpha \int f(x)x^{1-\alpha}\log x\,dx$$

3. $\{\cos kx, \sin kx\}$ is a basis of the homogenous equation. $W(\cos kx, \sin kx) = k$. So the Green's function is $G(x,t) = \frac{1}{k}(\cos kx \sin kt - \cos kt \sin kx) = \frac{1}{k}\sin k(x-t)$.

4. (a) $(D-\alpha)^{m+1}$, $(D^2+\beta^2)^{m+1}$, $D^2 - 2\alpha D + (\alpha^2+\beta^2)$, respectively.

   (b) $\{\cos x, \sin x, e^{3x}, xe^{3x}\}$ is a basis of solutions of the homogenous equation. An annihilator for $e^{3x}(10x+1)$ is $(D-3)^2$. So, design a particular solution as

$$c_1\cos x + c_2\sin x + (c_3 + c_4x + c_5x^2 + c_6x^3)e^{3x}$$

   A straightforward calculation shows that $u_p(x) = x^2(2x-3)e^{3x}$. Hence, the general solution is

$$u = c_1\cos x + c_2\sin x + c_3e^{3x} + c_4xe^{3x} + x^2(2x-3)e^{3x}$$

5. (a) $p(\lambda)$ is factorized in the real field into $\lambda + a$ and $\lambda^2 + p\lambda + q$, $(a, p, q \in \mathbb{R})$. Since the equation is asymptotically stable, a, p, q are all positive. This implies that all coefficients of the differential equations are positive.

   (b) Suppose it is <u>not</u> asymptotically stable. That means $a \geq 0$ is a root of the characteristic polynomial.

$$p(a) = a^n + a_1 a^{n-1} + \ldots + a_n \geq a_n > 0$$

   which leads to a contradiction.

   (c) A counterexample is $u''' + u'' + u' + u = 0$.

6. (a) $\dfrac{|y^2 - 0|}{|y - 0|} = |y| \to \infty$ as $y \to \infty$. So, not Lipshitzian. The solution of $\begin{cases} y' = y^2 \\ y(0) = y_0 > 0 \end{cases}$ is $y = \dfrac{y_0}{1 - ty_0} \to \infty$ as $t \to \frac{1}{y_0}$.

   (b) $\dfrac{|y^{2/3} - 0|}{|y - 0|} = |y^{-1/3}| \to \infty$ as $y \to 0$. So, not Lipschitzian. The initial value problem $\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases}$ has two solutions, $y_1(t) = 0$ and $y_2(t) = (t/3)^3$.