Part I Problems and Solutions

In the following two problems, find a fundamental matrix of the given system and then use it to find the specific solution of the system which satisfies the given initial condition.

Problem 1: $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Solution: We use the formula

$$\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$$

where $\Phi(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, to find the solution vector $\mathbf{x}(t)$ that satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

The answer is $\Phi(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$, and

$$\mathbf{x}(t) = \frac{1}{2} \begin{bmatrix} 5e^t + e^{3t} \\ -5e^t = e^{3t} \end{bmatrix}$$

Problem 2: $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution: We again use the formula

$$\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$$

where Φ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, to find the solution vector $\mathbf{x}(t)$ that satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

The answer is $\Phi(t) = \begin{bmatrix} 5\cos 4t & -5\sin 4t \\ 2\cos 4t + 4\sin 4t & 4\cos 4t - 2\sin 4t \end{bmatrix}$, and

$$\mathbf{x}(t) = \frac{1}{4} \begin{bmatrix} -5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{bmatrix}$$