Example

Remark on complex numbers. As we saw in the session on Complex Arithmetic and Exponentials in Unit I, the formula

$$D\left(c\,e^{at}\right) \;=\; c\,a\,e^{at} \tag{*}$$

remains true even when c and a are complex numbers. Therefore the rules and arguments above remain valid even when the exponents and coefficients are complex. We illustrate this with the following example.

Example. Find $D^3 e^{-t}\sin t$.

Solution using the exponential-shift rule. Using the exponential shift rule and the binomial theorem,

$$
\begin{aligned}
D^3 e^{-t}\sin t \;&=\; e^{-t}(D-1)^3\sin t \;=\; e^{-t}(D^3 - 3D^2 + 3D - 1)\sin t \\
&=\; e^{-t}(2\cos t + 2\sin t),
\end{aligned}
$$

since $D^2\sin t = -\sin t$ and $D^3\sin t = -\cos t$.

Solution using the substitution rule. Write $e^{-t}\sin t \;=\; \Im e^{(-1+i)t}$. We have

$$
\begin{aligned}
D^3 e^{(-1+i)t} \;&=\; (-1+i)^3 e^{(-1+i)t}, \qquad \text{by the substitution rule and (*);} \\
&=\; (2+2i)\,e^{-t}(\cos t + i\sin t),
\end{aligned}
$$

by the binomial theorem and Euler's formula. To get the answer we take the imaginary part: $e^{-t}(2\cos t + 2\sin t)$.

The operator method combined with the Exponential Response formula gives an efficient way to write and solve inhomogeneous DE's with real or complex exponential input. The following example again illustrates the usefulness of complex exponentials.

18.03SC Differential Equations

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