The Exponential Response Formula: Resonant Case

The starting point for understanding the mathematics of pure resonance is the generalized Exponential Response formula. First recall the simple case of the Exponential Response formula:
A solution to

$$p(D)x = Be^{at} \tag{1}$$

is given by

$$x_p = \frac{B\,e^{at}}{p(a)} \qquad \text{provided that } p(a) \neq 0. \tag{2}$$

In the session on Exponential Response we also saw the generalization of this formula when $p(a) = 0$. Here we will need to use the special case when $p'(a) \neq 0$: A solution to equation (1) is given by

$$x_p = \frac{B\,t\,e^{at}}{p'(a)} \qquad \text{if } p(a) = 0 \text{ and } p'(a) \neq 0 \tag{3}$$

We will call this the **Resonant Response Formula**.

Let's look at an example of the type we will be using here to study pure resonance.

Example. Find a particular solution to the DE $x'' + 4x = 2\cos 2t$.

As usual, we try complex replacement and the ERF: if z_p is a solution to the complex DE $z'' + 4z = 2e^{2it}$, then $x_p = \text{Re}(z_p)$ will be a solution to $x'' + 4x = 2\cos 2t$. The characteristic polynomial is $p(s) = s^2 + 4$, and $a = 2i$, so that we have $p(a) = 0$. But since $p'(s) = 2s$, we have $p'(a) = p'(2i) = 4i \neq 0$. The resonant case of the ERF thus gives

$$z_p = \frac{2\,t\,e^{2it}}{4\,i}.$$

Then taking the real part of z_p gives us our particular solution

$$x_p = \frac{1}{2}\,t\,\sin 2t.$$