Linear Differential Equations

1. Linear Differential Equations

A **linear differential equation** is of the following form:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 \dot{x} + a_0 x = q(t). \tag{1}$$

The a_k's are the *coefficients*. They may depend upon t (but not on x). If a_n is not zero then the differential equation is said to be of order n.

If this models a physical system then the left hand side represents the *system* and the right hand side represents the *input signal*. The coefficients represent parameters of the system. For example, the mass, damping and spring constants m, b and k in $m\ddot{x} + b\dot{x} + kx$ are the parameters of the system. In general, they may depend on time, e.g. maybe the force is actually a rocket, and the fuel burns so m decreases. Or maybe the spring gets softer as it ages. Maybe the honey in the dashpot gets thicker with time.

We will generally assume the coefficients are constant. In which case equation (1) is said to be a **constant coefficient linear equation**. It is, in fact, a good approximation of the non-constant coefficient equation as long as the coefficients vary on a time-scale that is much greater than the time-scale of the dynamical variable x.

2. Second Order Homogeneous Constant Coefficient Linear Equations

We will study the spring system $m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}$ starting with the case $F_{\text{ext}} = 0$.

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{2}$$

To ensure that (2) is of second order (and a realistic physical system) we always assume $m > 0$, but we will allow the case $b = 0$ and occasionally $k = 0$. With no external force the system evolves on its own. Think of a door that can swing back and forth or a ball on the end of a rubber band. As we did in first order equations we will call (2) a *homogeneous* linear differential equation.

3. The Undamped Case

The special case $b = 0$ is called **undamped**. This is called the **simple harmonic oscillator**. It's ODE is $m\ddot{x} + kx = 0$ or

$$\ddot{x} + \frac{k}{m} x = 0.$$

If we let $\omega = \sqrt{k/m}$ our equation becomes

$$\ddot{x} + \omega^2 x = 0.$$

We have seen before (and you can easily check) that $x_1(t) = \cos(\omega t)$ and $x_2(t) = \sin(\omega t)$ are solutions to this equation. Since the input is 0 and the equation is linear, we can use superposition of solutions to get the general solution

$$x(t) = a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi) \tag{3}$$

This is *another fundamental fact you should memorize!* (The second equality comes from the *sinusoidal identity*, which gives
$a = A\cos\phi$ and $b = A\sin\phi$.)

We know (3) gives every solution because $x(0) = a$ and $\dot{x}(0) = \omega b$, so you can solve (uniquely) for a and b to give any desired intial condition.