

Linear vs. Nonlinear

The general *linear* first order ODE is

$$r(t)x'(t) + p(t)x(t) = q(t).$$

The general first order ODE is

$$\dot{y}(t) = F(t, y(t)).$$

Representing the linear side of the debate is Linn E. R. and Chao S. representing the nonlinear side.

1. The Debate

Linn: I'd like to begin by making the point that there is a solution procedure for linear equations, which reduces the solution of any linear equation to integration. Multiply the equation through by a factor so that the two terms become the two terms in $\frac{d(ux)}{dt}$, then integrate.

Sometimes you can just *see* this. For example,

$$t^2\dot{x} + 2tx = \frac{d}{dt}(t^2x).$$

If we are in *reduced standard* form, i.e. when $r = 1$, then this can be done systematically with the following steps:

1. We seek $u(t)$ such that

$$u(\dot{x} + px) = \frac{d(ux)}{dt},$$

i.e. $pu = \dot{u}$. This is separable, with solution

$$u = e^{\int p(t) dt}.$$

(Any constant of integration will do here.)

2. Then integrate both sides of

$$\frac{d(ux)}{dt} = uq$$

and solve for x . This gives

$$x(t) = u^{-1}(t) \int u(t)q(t) dt.$$

The constant of integration is in this integral, so the general solution has the form

$$x(t) = x_p(t) + cu^{-1}(t).$$

Another lovely feature of linear equations is that the constant of integration in the solution of a linear equation always appears right there.

The *associated homogeneous equation* is

$$\dot{x} + px = 0.$$

This is separable, with solution $x_h(t) = e^{-\int p(t) dt}$. Look! This is the reciprocal of the integrating factor! Wonderful!

In most applications, $u^{-1}(t)$ falls off to zero as t gets large; the term $cu^{-1}(t)$ is a *transient*.

Chao: That's a lot of integration. I'm more interested in the general behavior of solutions, rather than an incomprehensible expression of them as integrals or a boring expression of them in terms of sin, cos, and exponentials.

I prefer arguments like this: take an equation like

$$\dot{y} = y^2 - x.$$

This doesn't have a single solution which you, Linn, in your linear cave, have anything to say about. But I can look at the direction field, recognize that there is a funnel along the curve $y = -\sqrt{x}$. This means all solutions near there are trapped and are asymptotic to $-\sqrt{x}$. I can even argue that they are all ultimately a bit larger than $-\sqrt{x}$. No integration involved and very good information.

Or take an equation like the logistic equation

$$\dot{y} = k_0y \left(1 - \frac{y}{p}\right).$$

This is an autonomous equation, and remains so even if I allow a harvest rate, even one depending upon y :

$$\dot{y} = k_0y \left(1 - \frac{y}{p}\right) - a(y).$$

This equation gives genuine insight into real population dynamics. By looking at the phase line it is easy to analyze the behavior of solutions, in a way useful for policy makers.

Linn: Well, now, most of the time a system is near equilibrium. Engineers get very anxious when their systems get too far from equilibrium. Let's look at your nice nonlinear logistic equation. As you said, there's a critical point at $y = p$, and so an equilibrium solution. Just how does the system relax to this equilibrium?

Let's write $y = p + u$ and change variables using

$$1 - \frac{y}{p} = -\frac{u}{p}.$$

Substituting this in the logistic equation gives

$$\dot{u} = -k_0 \frac{u}{p} (p - u) = -k_0 u + k_0 \frac{u^2}{p}.$$

For small u the second term is *very* small, and can be ignored. This is called **LINEARIZING!!** the equation near equilibrium. Near equilibrium solutions to the nonlinear equation behave a lot like $p + u$ where u is a solution to the *linear* equation

$$\dot{u} = -k_0 u.$$

So we can say that the population relaxes to equilibrium exponentially, as $e^{-k_0 t}$ approaches 0.

Chao: There you go again with your fancy exponentials. You think you know all about them, but your computer has to compute their values, after all, and the methods it uses are no different from the methods used to compute the values of linear equations. It has to use Euler's method, or its fancier variants.

Linn: Speaking of fancy exponentials, I'd like to point out that smart people almost never use integrating factors to integrate linear equations with constant coefficients:

$$\dot{x} + kx = q(t). \tag{LCC}$$

Yeah, the integrating factor is e^{kt} , but even I don't like the integrals that come out. But there are these great tricks, Chao! Suppose $q(t) = Be^{rt}$. Then, be optimistic! Maybe there's an exponential solution of the form Ae^{rt} . When we substitute into the two sides of equation (LCC) we get:

$$\text{Left side: } \dot{x} + kx = A(r + k)e^{rt}$$

$$\text{Right side: } Be^{rt}.$$

Setting them equal to each other we get:

$$Be^{rt} = A(k + r)e^{rt}$$

Thus,

$$A = \frac{B}{k+r} \quad \text{and} \quad x_p(t) = \frac{B}{k+r} e^{rt} \quad (\text{ERF})$$

We've found a particular solution to $\dot{x} + kx = Be^{rt}$. Because it is the response to exponential input we call this the Exponential Response Formula. (ERF) It works as long as $k+r$ is not 0.

Chao: Bravo.

Linn: Thank you. And what's better, did I ever tell you about the complex exponential? In the ERF r can be a complex number! Euler told us that

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

– it's a point on the unit circle in the complex plane. So, trig functions are incorporated into the complex exponential!! To solve

$$\dot{x} - x = 3 \cos(t)$$

I replace it by the different equation

$$\dot{z} - z = 3e^{it}$$

of which my original DE is the real part. Then I can use the ERF (with $k = -1, r = i, B = 3$) to get

$$z_p = 3 \frac{e^{it}}{(-1+i)}.$$

Since $x_p = \text{Re}(z_p)$, all that's left is to find the real part of z_p .

$$z_p = 3 \frac{e^{it}}{(-1+i)} \frac{-1-i}{-1-i} = \frac{-3 \cos(t) + 3 \sin(t) + i(-\cos(t) - \sin(t))}{2}.$$

Therefore,

$$x_p = \text{Re}(z_p) = -\frac{3}{2} \cos(t) + \frac{3}{2} \sin(t).$$

For the general solution we just add in the general solution to the homogeneous equation, which is ce^t . It's a little funny to call this a *transient* and I don't, but it does give you the general solution.

Chao: You know, your method results in these sums of sines and cosines, which is very nice but I want to know what they look like. The *nonlinear* view of sines and cosines writes

$$a \cos(\omega t) + b \sin(\omega t) = A \cos(\omega t - \phi),$$

where A , and ϕ are the polar coordinates of the point (a, b) .

In your example, $A = 3\sqrt{2}/2$ and $\phi = 3\pi/4$. So, the solution is easy to draw and compare with the input signal.

I want to point out another charming feature of solutions to many nonlinear equations. Take a simple one, for example $\dot{y} = y^2$. This is separable and can be solved in three short steps:

$$\begin{aligned} y^{-2} dy &= dx \\ -y^{-1} &= x + c \\ y &= 1/(c - x). \end{aligned}$$

So the IVP with $y(0) = 1$ has $c = 1$ and $y = 1/(1 - x)$. Its graph is asymptotic to the vertical line at $x = 1$. In other words, it is able to go off to infinity in finite time. It *ends*. The equation $y = 1/(1 - x)$ actually represents *two* solutions: one for $x < 1$, and another for $x > 1$. If we are to say that a solution of a differential equation is determined by an initial value, we have to require that the graph be connected.

Linn: You call that a feature? That never happens to solutions to my equations. If they are going to go south on me, I know it from the coefficients or the input signal. As long as $p(t)$ and $q(t)$ are nice and finite (and $r(t)$ is nonzero) so are all solutions. They live as long as I do!

Chao: Well, the world really is nonlinear. Newton's law of gravitation is highly nonlinear. This kind of explosion actually happens in the case of Newton's laws: Jeff Xia showed that in a certain 5-planet system two of the planets behaves more or less like $\frac{\sin(1/t)}{t}$, oscillating with increasing amplitude and increasing frequency as $t \rightarrow 0$ (from the negative side).

Solutions to linear equations are not nearly as diverse and exciting!

2. Conclusion

In both 18.03 classes in spring 2010, Linear won the debate, but Nonlinear's supporters were more enthusiastic.

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