Superposition Principle

1. Superposition Principle for Inputs

We conclude our introduction to first order linear equations by discussing the *superposition principle.* This is the most important property of these equations. In fact, we will see in later sessions that superposition is the defining characteristic of linear equations of any order.

In the examples below we will see that the superposition principle allows us to break up a problem into simpler problems and then at the end assemble the answer from its simpler pieces.

As usual in this session we are considering the first order linear equation

$$\dot{y} + p(t)y = q(t).$$

We will consider the left-hand side to be the system and the right-hand side to be the input.

For any given input $q(t)$ that has output $y(t)$ we will write $q \rightsquigarrow y$ (read input q leads to output y).

If $q_1(t)$ and $q_2(t)$ are signals, and c_1 and c_2 are constants then we call $c_1q_1(t) + c_2q_2(t)$ a **superposition** of q_1 and q_2. Another name we will often use for this is a **linear combination** of q_1 and q_2.

Superposition Principle: For constants c_1 and c_2

$$\text{If } q_1 \rightsquigarrow y_1 \text{ and } q_2 \rightsquigarrow y_2 \quad \text{then} \quad c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2.$$

Proof: This is true because the ODE is linear. The proof takes two lines:

$$\begin{aligned} \frac{d(c_1y_1 + c_2y_2)}{dt} + p(c_1y_1 + c_2y_2) &= c_1(\dot{y}_1 + py_1) + c_2(\dot{y}_2 + py_2) \\ &= c_1q_1 + c_2q_2. \end{aligned}$$

We present some easy examples below.

2. Examples

In the next session we will learn how to find the solutions to the following ODE's . For now, you can check the given solutions by substitution into the DE.

i. $\dot{x} + 2x = 1$ has a solution $x(t) = \frac{1}{2}$

ii. $\dot{x} + 2x = e^{-2t}$ has a solution $x(t) = te^{-2t}$

iii. $\dot{x} + 2x = 0$ has a solution $x(t) = e^{-2t}$.

Using the solutions above as a basis, we can solve more complicated equations.

Example 1. Use superposition to find a solution to $\dot{x} + 2x = 1 + e^{-2t}$

Solution. The input is a superposition of the inputs from (i) and (ii). Therefore a solution is $x(t) = \frac{1}{2} + te^{-2t}$.

Example 2. Find a solution to $\dot{x} + 2x = 2 + 3e^{-2t}$.

Solution. The input is $2 \cdot (1) + 3 \cdot (e^{-2t})$; it is a superposition (with coefficients $c_1 = 2$, $c_2 = 3$) of the inputs from (i) and (ii). Therefore, $x(t) = 2 \cdot \frac{1}{2} + 3(te^{-2t}) = 1 + 3te^{-2t}$ is a solution.

Example 3. Find *lots* of solutions to $\dot{x} + 2x = 1$

Solution. We can write the input as:

$$1 = 1 + c \cdot 0.$$

That is, as a superposition of the input from (i) and the homogeneous equation (iii). Therefore $x(t) = \frac{1}{2} + ce^{-2t}$ is a solution for any value of the parameter c.

Example 4. Find *lots* of solutions to $\dot{x} + 2x = 1 + e^{-2t}$.

Solution. Use superposition: $x(t) = \frac{1}{2} + te^{-2t} + ce^{-2t}$.