PROFESSOR: Welcome to this recitation on matrix exponential. So here, we're given matrix A with entries 6, 5, 1, 2. And we're asked to compute the matrix exponential, exponential A*t, and to use it to solve the initial value problem u prime of t equals A*u(t), where here u are basically vectors, with initial condition, u of 0 equals [4, 1]. So why don't you pause the video, work through the problem? And I'll be right back.

Welcome back. So first, to go ahead and compute the matrix exponential, we need to identify the eigenvalues of matrix A and its eigenvectors. So this is a matrix-- I'll just rewrite here-- that we saw before. And its eigenvalues are again, solution of 6 minus lambda, 5, 1, 2 minus lambda, equals to 0, which gives us 6..., 2 minus lambda minus 5 equals to 0. Lambda square minus 8*lambda. Then we have a 12 minus 5. So you can verify that the two eigenvalues would be 1 and 7. Lambda_1 equals to 1. And lambda_1 equals to 7.

So now, we need to seek the eigenvectors associated to each one of the eigenvalues. So the idea here is to basically move toward a diagonalization of the matrix A. So let's seek the eigenvectors. And here, I'm just going to give them to you, and you can verify the calculation. And this calculation was performed in a previous recitation.

So the eigenvectors. v_1 associated to the eigenvalue lambda_1 was, for example, 1, minus 1. And the other one that we found-- again, this is one form of the eigenvector-- was 5 and 1. So these are from the notes of a previous recitation. So you can verify that these are the two eigenvectors.

And from this point, then we can rewrite this solution, if you recall. I'm just going to go through the steps toward getting to the definition of the exponential matrix. So here, if we didn't know anything about the exponential matrix, we would be able to write the solution as c_1 exponential t v_1 plus c_2 exponential 7t v_2, which basically gives us here, if I write it in this form, for example, an exponential t, minus exponential t and an exponential 5t multiplied by the entry of this vector, an exponential 7t here, multiplying [c_1, c_2].

So this is where the idea of the matrix exponential comes from. We're basically introducing the matrix phi of t for which we can write u equals phi of t multiplied by this [c_1, c_2], general constants. So phi of t would then be equal to this matrix. But what we want is to be able to solve an initial value problem for which e of A of 0 applied to our initial conditions would give us back our initial condition. So we're seeking for a form for this exponential matrix that would allow us to do this.

So the way that we define the matrix exponential give us exponential A*t-- now, I won't go into the proof, but we're just going to check it together-- multiplied by phi of 0 minus 1. So let's check that if we use this form of the matrix exponential, we would have e. We will have that at 0 applied to u(0). We have phi(0), phi(0)^(-1) applied to u(0). This is a matrix with its inverse, which gives us the identity. And so basically, this gives us back u of 0.

I mean you don't need to do that when you're asked to find the matrix exponential. But just to remember where it's coming from, you write down your system in matrix form. You identify the matrix phi of t. And then you recall why you want the matrix exponential to have this form, basically to be able to solve initial value problems for which the value u of t is projected to u of 0 when we take t equals 0 for the matrix exponential.

So now let's go back to our problem. So let's compute this matrix exponential. We have phi of t. So now from this formula, we know that we need phi of 0. So that give us, basically, exponential of 0, 5, minus 1, and 1. We need to find its inverse. So recall that the inverse of a two-by-two matrix is basically just the determinant, minus b, minus c, and reversing the diagonal entries. So we can just apply this to get our phi of 0 minus 1. So here, our determinant is basically 1 plus 5, which is 1 over 6. And then the entries are simply 1, 1, minus 5, and 1.

So now, we're just left with the multiplication of two matrices to get our matrix exponential. So our matrix exponential would give us this one sixth. And we now have to multiply the entries. So I'm not going to rewrite everything. I'm just going to use this space here. So we have exponential t multiplying 1, plus 5 exponential 7t. Then, we have exponential t dot minus 5 for this entry. 5 exponential t multiplying our 1. 7t, thank you. Then for the second entry, we basically have minus exponential t 1 exponential 7t 1 minus exponential t minus 5 and exponential 7t 1.

So we're done with the matrix exponential. So now we were asked to solve for the initial value problem with initial condition 4 and 1. So how do we go about that? Well, recall that I just reminded you what did we want to use this matrix exponential for. And what we wanted it for is to be able to basically project an initial condition into a solution u of t, t times later. And we constructed this matrix to be able to basically give us this solution by just multiplying the matrix by the initial value vector.

So basically, to find the solution of this initial value problem, we simply need to multiply this matrix by the initial vector that we were given. And I'm just going to write it here to not have to rewrite everything. And it was 4 and 1. And this is u of 0. So let me just do a dash here just so that we can do the computation. And we would end up with a solution-- I'm going to keep it in matrix form for now.

So we end up with 4 exponential t minus 5 exponential t, so minus 1 exponential t. And we have a one sixth. Here, 5 exponential 7t, so we have 20, plus 5, so 25, exponential 7t. Then for the second entry of the vector solution, we have minus exponential here minus 4 that we add to a 5, and here, a 7 multiplied by 4 that we add to a 1. So we have basically plus 5 exponential 7t. And that basically gives us one way of writing this solution.

And we can split this down, if we will, into two vectors, plus t; minus 1, 1; exponential 7t; [25, 5]. And this form is as valid. Yes, thank you. So that ends the laborious calculations. But basically, the key point here was just to remember where is the matrix exponential coming from, basically, from the eigenvalues and eigenvectors of the original matrix present in the system, and where is the definition coming from, why do we define it as phi of t phi minus 1 of 0, and how to use it then to give the solution to an initial value problem. So that ends this recitation.