18.03SC Practice Problems 1

Introduction, natural growth and decay, review of logarithm

Solution Suggestions

An African government is trying to come up with a good policy regarding the hunting of oryx. They are using the following model: the oryx population has a natural growth rate of k years^{-1}, and there is assumed a constant harvesting rate of a oryxes/year.

1. *Write down a model for the oryx population. [First step: choose symbols and units.]*

Begin by identifying the variables that appear in the problem.

We are looking for a model for the size of a population of oryxes, which is changing over time. Here there are two variables: number of oryxes and time passed since some conveniently chosen initial point (e.g., since the time the government first started monitoring the size of the population). Choose units to reflect the terms given in the problem statement. The constants a and k were given in units oryxes/year and 1/years, respectively, so measure time in years and size of the population in number of oryxes. Fixing notation, say that at time t years, the oryx population is x oryxes, where $x = x(t)$.

The goal of the problem is to write down a relationship between these two variables as a mathematical expression. Here the intended model is a differential equation.

The first statement that the "natural growth rate of x is k" means that the oryx population increases over time at a rate proportional to the current size of the population with coefficient of proportionality k. This means that at any time t, after some small amount of time Δt years pass, we expect the number of oryxes to increase by approximately $kx(t)\Delta t$, provided there is no external interference. However, as the second statement says, here we also have to take into account external interference in the form of a "constant harvest rate of a" - that is, a harvesting of oryxes at a rate that is the same over time, independent of the current size of the oryx population. This means that at time t, after Δt years pass, the population also decreases by $a\Delta t$ oryxes because of harvesting. Combining these two statements gives that the total change in population at time t over a small interval Δt is approximately

$$x(t + \Delta t) - x(t) \simeq kx(t)\Delta t - a\Delta t.$$

Dividing both sides by Δt and letting Δt approach 0, we obtain the differential equation

$$\frac{dx}{dt} = kx - a,$$

which gives the desired model for the oryx population.

2. *Suppose $a = 0$. What is the doubling time?*

When $a = 0$, the differential equation becomes $\frac{dx}{dt} = kx$.

The doubling time for natural growth/decay systems is defined as the time it takes for the population to double. This cannot be read off directly from the infinitesimal information afforded by the differential equation, as it requires knowledge of the long-term behavior of the system. So we need to find an explicit equation for $x = x(t)$.

Solve the differential equation for $x(t)$. Separate variables

$$\frac{dx}{x} = kdt,$$

and take the integral of both sides to obtain

$$\ln |x| + c_1 = kt + c_2.$$

[Recall that $\int \frac{1}{x} dx$ is the natural log of the absolute value of x, up to a constant of integration. The absolute value is often ignored for convenience, but we will need it here.] Finally, combine constants into a single constant c on the right hand side

$$\ln |x| = kt + c,$$

and exponentiate both sides to get an explicit equation for $x = x(t)$,

$$x\,(t) = \pm e^c e^{kt},$$

after unwrapping the absolute value function.

A caveat: the coefficient here, $\pm e^c$, can be any nonzero number. When we first separated variables and divided by x, we had to implicitly assume that x was nonzero. Thus separation of variables gives only the nonzero solutions. The constant identically-zero solution $x \equiv 0$ also satisfies the differential equation, and is missing from the answer we got. Solving by separating variables usually has such "missing solutions". Taking into account this missing solution $x \equiv 0$, we can summarize that the general solution to $\frac{dx}{x} = kx$ is given by $x\,(t) = Ce^{kt}$, where C is any constant. Since $x\,(0) = Ce^{k \cdot 0} = C \cdot 1$, C is the "initial value".

By definition, the doubling time is the time T for which $x(T) = 2x(0) = 2C$. Using the expression we have found for $x(t)$, this equation becomes $e^{kT} = 2$, provided C is not 0. Hence, the doubling time is

$$T = \frac{\ln 2}{k},$$

provided the natural rate of growth k and the initial number of oryxes are both positive, which is sort of the underlying assumption in the way the problem is phrased. Both of these conditions make sense. The term "doubling time" is only used when the rate of growth is positive. If the rate of growth were negative, this would be a system with a positive "natural rate of decay" $-k$, and a defined concept of a "half-life" - the time for the population to decrease by half. The above formula still holds in that case, and gives a negative value for T. But then, as can be checked, the positive number $-T$ gives the half-life, which is reassuring. Also,

if the rate of growth/decay were zero, or if the population were identically zero, concepts of doubling time/half life do not make sense and are not defined.

Finally, it should be remarked that it is okay that the doubling time uses the implicit, potentially arbitrary, choice made of initial time $t = 0$ because the time it takes for the population of a natural growth system to double does not depend on count-off time. For example, here you can check that $x(t + T) = 2x(t)$ for all t. That is, the doubling time is well defined.

3. *Find the general solution of this equation.*

We are solving $\frac{dx}{dt} = kx - a$ for arbitrary a, k. Break down into cases, based on the value of k.

If $k = 0$, the equation is $\frac{dx}{dt} = -a$, with solution $x(t) = -at + C$, for some constant C. Here the initial value is $x(0) = C$.

Now suppose $k \neq 0$. We will solve by separating variables and then doing a substitution of variables to integrate. For convenience, we choose to leave the nonzero constant k on the right-hand side.

$$\frac{dx}{x - a/k} = kdt$$

Integrate by a substitution of variables. Let $u = x - a/k$. Then $\frac{du}{dt} = \frac{dx}{dt}$, and the equation becomes $\frac{du}{u} = kdt$. As discussed in the previous part, this has solutions $u(t) = Ce^{kt}$. Substituting back in for u, we find that

$$x(t) = Ce^{kt} + \frac{a}{k}.$$

The initial value is $x(0) = C + a/k$. Note that the missing solution here is $x \equiv a/k$.

4. *Check that the proposed solution satisfies the ODE.*

The solution was in divided in cases. Check each.

For $k = 0$, $\frac{dx}{dt} = \frac{d}{dt}(-at + C) = -a = kx - a$.

For $k \neq 0$, $\frac{dx}{dt} = \frac{d}{dt}(Ce^{kt} + \frac{a}{k}) = Cke^{kt} = k(Ce^{kt} + a/k) - a = kx - a$.

5. *There is a constant solution. Find it. Does the way the solution depends upon k and a make sense? That is, do the units come out right? Does it behave right when a is large or small? When k is large or small?*

A constant solution is one for which $\frac{dx}{dt} \equiv 0$. Using our differential equation, $\frac{dx}{dt} = kx - a$, we get that a constant solution is one with $kx = a$.

If $k = 0$, this is only possible when $a = 0$, i.e., in the absence of harvesting, and then all solutions are constant.

If $k \neq 0$, we can divide by k to see that x must equal a/k identically, and obtain that the only constant solution is $\frac{dx}{dt} = kx - a$. An alternative way to proceed in this case would be to notice that the condition for a constant solution together with our knowledge of the form of the general solution in this case gives the condition $Cke^{kt} = \frac{dx}{dt} \equiv 0$. An exponential is never zero, and we are assuming k is not zero, so C must be zero. Plugging this into the general solution gives $x(t) = a/k$ for all times t. The first method does not use knowledge of the general solution and shows that here we could find constant solutions from just the infitesimal data given by the differential equation. The second method gives some intuition for what is happening on the global level, and serves as a check of the steps we've done.

Finally, we should be able to check that the constant solution we found in the non-degenerate case, $x(t) = a/k$, makes sense in the context of the probem. First, a basic check on units shows that both sides are measured in number of oryxes. Furthermore, we can rewrite this equation as $kx = a$. The population we are modelling is infinitesimally increasing by kx oryxes per year because of the natural growth and decreasing by a oryxes per year because of harvesting. Intuitively, the population should hold steady when the rate of increase is equal to the rate of decrease, which is the equation we have obtained. So the way the constant solution depends on k and a does make sense.

6. *Graph the constant solution (equilibrium) and some others, and comment on what they signify. Why, in this case, do we say the constant solution is "unstable?" Notice that for initial values less than the equlibrium, the solutions stop having meaning in terms of the real-world situation they are modeling when they become negative (of course, this is true for all initial values, but for somewhat different reasons). In these cases, predict the time t_e at which oryxes will become extinct. For example, suppose that x_0, the population of oryxes at time $t = 0$, is half the equilibrium. For this initial condition what is t_e? Your answer should depend on k.*

The solution for this problem is written for the intended scenario - the population we are modeling has natural growth rate and harvesting rate both positive. In other situations, either k or a or both could have been negative, and it is educational to consider what happens in those cases.

Here assume $a, k > 0$. For initial oryx populations above the equilibrium a/k, harvesting does not cancel out natural growth, and the popluation increases without

bound. For initial populations below a/k, harvesting overtakes the growth rate, and the population goes to zero. Thus the constant solution is unstable. Let x_0 denote the initial population. Determine the extinction time t_e as follows. First of all, $x_0 = x(0) = Ce^{0k} + a/k = C + a/k$, so $C = x_0 - a/k$, which is negative for $x_0 < a/k$. Solving

$$0 = x(t_e) = (x_0 - a/k)e^{kt_e} + a/k$$

for t_e, obtain

$$t_e = \frac{1}{k} \ln \frac{a}{a - kx_0},$$

provided both a and k are positive. In particular, when $x_0 = \frac{a}{2k}$, t_0 is given by $\frac{\ln 2}{k}$.

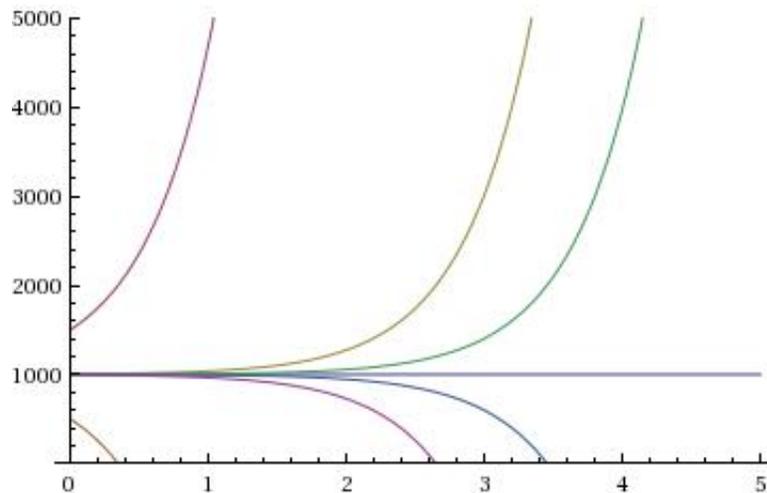Below is the graph of the constant solution and some others for $k = 2$, $a = 2000$.

Figure 1: Plot of $x(t) = Ce^{2t} + 1000$ for $C = -500, -5, -1, 0, 1, 5, 500$

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