What Can Go Wrong

If the homogeneous DE $p(D)y = 0$ has polynomial solutions, then the polynomial solution of the inhomogeneous DE $p(D)y = q$ will be of higher degree than the degree of $q(x)$. We illustrate with an example.

Example. Solve $y'' + y' = x + 1$

Try $y_p = Ax + B \Rightarrow 0 + A = x + 1$ –can't solve.

Problem: the constant term in $y'' + ay' + b$ is 0.

Fix: bump all degrees up by order of lowest derivative: try $y_p = Ax^2 + Bx$.

Substitute: $2A + (2Ax + B) = x + 1$

Equate coeff: $2Ax + (2A + B) = x + 1 \Rightarrow A = 1/2, B = 0 \Rightarrow y_p = \frac{1}{2}x^2$.

Example. $y''' + 3y'' = x^2 + x$

Lowest order derivative is 2 $\Rightarrow$ bump up all degrees by 2. Try $y_p = Ax^4 + Bx^3 + Cx^2 \Rightarrow (24Ax + 6B) + 3(12Ax^2 + 6Bx + 2C) = x^2 + x$.

Equate coefficients: $36A = 1$, $24A + 18B = 1$, $6B + 6C = 0$ (we'll skip the algebra).