PROFESSOR: Hi everyone. Welcome back. So today, I'd like to tackle a problem in undetermined coefficients, specifically find a particular solution to each of the following equations using undetermined coefficients. So for part A, we have x dot plus 3x equals t squared plus t. And for B, we have x dot dot plus x dot equals t to the four. So I'll let you work this problem out. And I'll come back in a minute.

Hi Everyone. Welcome back. So we're asked to solve this problem using the method of undetermined coefficients. And specifically, the observation is if we have a differential equation with constant coefficients, and we have a forcing on the right-hand side which is a polynomial, then there's always going to be a particular solution, which is a polynomial, that has the form of some constant times t to the power of r plus constant $t^{(r-1)}$ plus a constant $c_{(r-2)} t^{(r-2)}$ plus dot, dot, dot, plus c_1*t and then possibly plus c_0.

And typically, the problem is to find out what's the highest-order polynomial that we should guess. These constant c's, they're referred to as the undetermined coefficients. And these are the constants that we seek to solve.

So for our first differential equation, we have x dot plus 3x equals t squared plus t. And the power that we should guess for a solution are, it's always going to be at least as high as the highest power on the right-hand side. So the right-hand side tells us at least the largest that we should at least start our guess at. And then, sometimes, we have to knock it up a few orders depending on what the lowest-order derivative on the left-hand side is.

So in this case, the term on the left-hand side, there's no derivative term. And so what we can do for this case is we can just take r is equal to 2, the power of the polynomial on the right-hand side. So we're going to seek a guess or an ansatz, which is some constant A times t squared plus B times t plus some constant C. And again, I've taken the highest polynomial that we should guess to be 2, because the right-hand side is 2 in this case.

So now what we do is we just substitute this in into the differential equation. And we choose our A, B, and C to construct this as a solution. So let's go ahead and do that. So taking its derivative, we have 2A*t plus B. And then, we have 3 times A*t squared plus B*t plus C. And we want this to be equal to t squared plus t.

And now, the only way that the left-hand side is going to equal the right-hand side is if the coefficients in front of each polynomial are the same. So in the left-hand side, we only have one term that has a t squared in it. And that must, therefore, equal the t squared term on the right-hand side. So these two terms have to be equal to each other.

So we end up getting that 3A has to be equal 1. And so A is equal to one third. And now what we do is we collect terms with a power of t. This gives us 3B plus 2A on the left-hand side, so that's this term and this term. And we want that t equal t on the right-hand side or just 1. Now, we already know that A is equal to one third. So we end up getting that 3B is equal to 1 minus 2/3, which is equal to one third. Put 3B as one third, which means that B is equal to 1/9.

And now lastly, to determine C, we see that B plus 3C must be 0. So B plus 3C is 0, or C is equal to negative 1/27. So notice how we always start with the highest power. In this case, A is the coefficient in front of t squared. That lets us solve for A. And then the rest of the undetermined coefficients, we can solve for, almost like a giant zipper. So the particular solution that we just constructed, when the dust settles is 1/3 t squared plus 1/9 t minus 1/27. So this concludes part A.

For part B, we have the differential equation x dot dot plus x dot equals t to the four. And in this case, we know that the right-hand side is a power of four. So we should guess at least a fourth-order polynomial. However, we see that there's no constant multiple of x. In fact, the lowest-order derivative is 1. So what this means is we should actually knock the solution that we guess up one order polynomial.

So we should try and guess x is equal to A t to the fifth plus B t to the four plus C t cubed plus D t squared plus E*t. And we can leave out an F term. We can leave out a constant. Because if we just substitute that in the left-hand side, we see that the x dot dot is going to vanish, and x dot is going to vanish. So we can omit any constant term.

So let's seek this ansatz, and plug it into the differential equation. And if we do that, we get 20A t cubed. And this is plugging into the x dot dot term. This term is going to give us 12B t squared plus 6C*t plus D. And then the x dot term is going to give us 5A t to the four plus 4B t cubed plus 3C t squared plus 2D*t plus E.

And we want the sum of these two terms to equal t to the four. So again, what we do is we start with the highest power. We see that on the left-hand side, we have 5A must be equated with just 1. So we have 5A is equal to 1, or A is equal to 1/5. Secondly, we're going to have

20A plus 4B equals the 0 polynomial, or the 0 which is the coefficient of t cubed. And this gives us B is equal to negative 5A, which is negative 1.

And now for the quadratic terms, we have 12B plus 3C is 0. So C is going to equal negative 4B, which is negative 4 times negative 1, which just gives us 4. The linear term is going to give us 6C plus 2D equals 0, which gives us D is equal to negative 3C, which gives us negative 12. And then the last term is D plus E is 0, which gives us E is equal to 12.

So at the end of the day, we end up with a particular solution, which is a polynomial of 1/5 t to the five minus 2 to the four plus C, which is 4, t cubed plus D, which is negative 12, t squared plus E, which is 12, times t. And by construction, this polynomial solves the differential equation with a forcing of t to the four on the right-hand side.

So just to recap, when we're faced with an undetermined coefficient problem, what we have to do is we just guess a solution, which is a polynomial. And the main difficulty is just guessing the highest power of the polynomial. And that's always going to be at least as big as the polynomial on the right-hand side. But as we saw in part B, we sometimes have to knock it up a few orders depending on what the differential equation actually is. So I'd like to conclude here. And I'll see you next time.