

Phase Lines

1. Definition

Since the direction field for an autonomous DE, $\dot{y} = f(y)$, is constant on horizontal lines, its essential content can be conveyed more efficiently using the following recipe:

1. Draw the y -axis as a vertical line and mark on it the equilibria, i.e. where $f(y) = 0$.
2. In each of the intervals delimited by the equilibria draw an upward pointing arrow if $f(y) > 0$ and a downward pointing arrow if $f(y) < 0$.

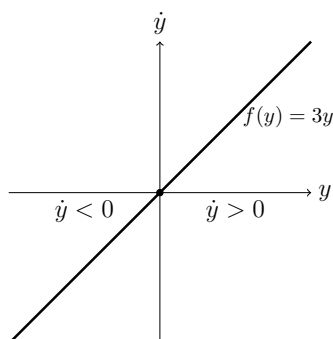
This simple diagram tells you roughly how the system behaves. It's called the **phase line**. The phase line captures exactly the information we use to get the qualitative sketch of solution curves. We illustrate this with some examples.

2. Examples

Example 1. For the DE $\dot{y} = 3y$: find the critical points, draw the phase line, classify the critical points by stability and use the phase line to give a qualitative sketch of some solution curves.

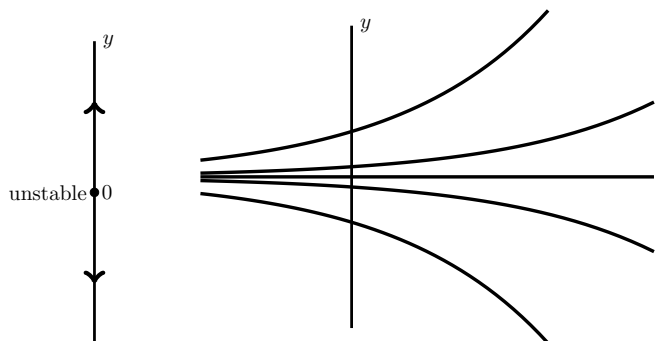
Solution. The steps to follow are:

1. Find the critical points.
 2. Plot the graph of $f(y)$ and determine where \dot{y} is positive and negative.
 3. Draw the phase line and find the stability of the critical points.
 4. Sketch the solution curves.
1. We have $f(y) = 3y$. We easily see that the only critical point (root of f) is $y = 0$.
 2. The plot of $f(y)$ is a straight line. We see $\dot{y} > 0$ for $y > 0$ and $\dot{y} < 0$ for $y < 0$.



3. We use the information from steps 1 and 2 to draw the phase line. We put a large dot at the critical point. Since $\dot{y} > 0$ in the interval $y > 0$ we add an arrow pointing upwards in that interval. Similarly the interval $y < 0$ gets a down arrow. Since the arrows point away from the critical point, the equilibrium is *unstable*. This is all shown in the figure below.

4. Once we have the phase line we can make a qualitative sketch of the solution curves. The equilibrium solution corresponds to the critical point. It is the horizontal line $y(t) = 0$. The solutions that start positive increase and those that start negative decrease. We present the solution curves next to the phase line so you can see that the phase line arrows represent the y -direction of the integral curve.

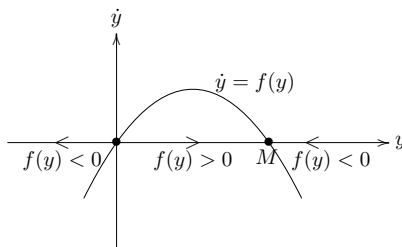
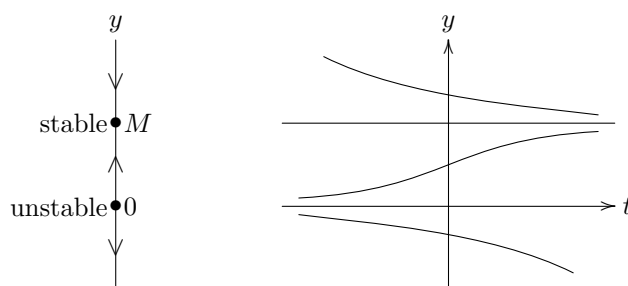


3. The phase line. 4. Qualitative sketch of solution curves.

Example 2. Repeat example 1 for the logistic equation $\dot{y} = k_0(1 - y/M)y$.

Solution. We did all of this earlier except draw the phase line.

1. Critical points: $y = 0, y = M$.

2. Graph of $f(y)$ 

3. Phase line. 4. Sketch of solution curves.

3. Solutions Can Be Shifted in Time

In an autonomous equation $\dot{y} = f(y)$, the direction field is constant in the horizontal direction. Said differently, the conditions represented by the ODE are constant in time. Consequently, any horizontal (time) translate of a solution is another solution. A "time translate" of a function $y(t)$ is a function $y(t - t_0)$; the graph is shifted horizontally (to the right) by t_0 units.

Example 3. Solutions of $\dot{y} = k_0 y$ exhibit three different behaviors, illustrated by

$$y(t) = e^{k_0 t}, \quad y(t) = 0 \quad \text{and} \quad y = -e^{k_0 t}.$$

Any solution is a horizontal translate of one of these three:

$$y(t) = e^{k_0(t-t_0)},$$

$$y(t) = 0 \quad (\text{whose only time-translate is itself}), \text{ or}$$

$$y(t) = -e^{k_0(t-t_0)}.$$

See the answer to example 1 for the graphical version of this.

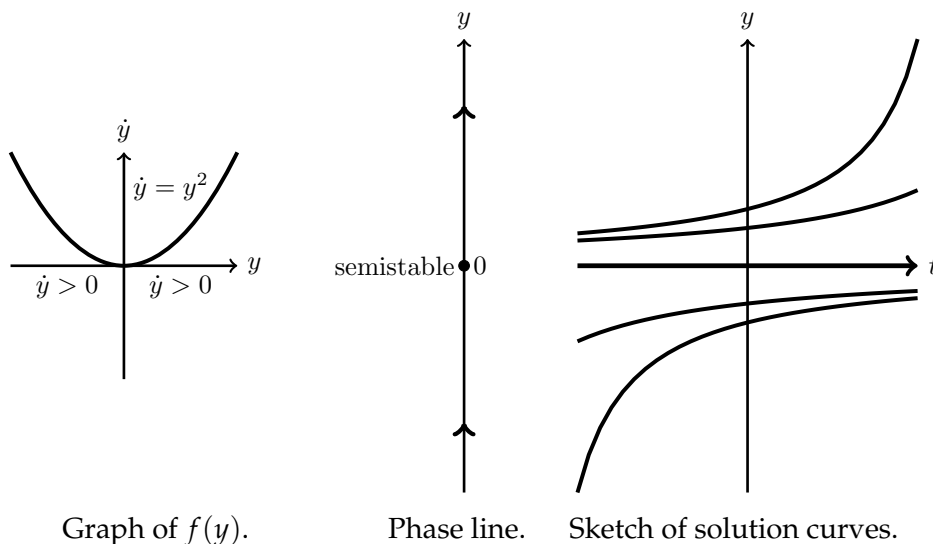
4. Semistable Equilibria

Some equilibria are half stable and half unstable. We call them **semistable**

Example 4. Repeat examples 1 and 2 for the DE $\dot{y} = y^2$.

Solution. Once again the algebra is trivial.

1. The only critical point is $y = 0$.
2. The graph of $f(y)$ shows \dot{y} is always positive (except at 0).
3. The phase line shows the critical point at 0 is *semistable*.



5. Conclusion

The phase line shows the qualitative behavior of a system at a glance: the critical points are shown and you can tell the stability of each critical point by looking at the arrows around it; the arrows also tell you what happens to the integral curves in the long-run, as t goes to infinity.

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