

PROFESSOR: Welcome to this recitation on pure resonance. So here we're given an operator $p(D)$ equals D square plus $4I$, where D is the differential operator and I is the identity operator. And you're asked to consider the equation $p(D)$ applied to x equals $F_0 \cos(\omega t)$, where F_0 is a constant. So the first question is what is the natural frequency of the system.

The second one is to use the exponential response formula to solve for $p(D)x$ equals $F_0 \cos(\omega t)$. And here you need to be careful and do it for both cases ω equal to 2 and ω equals not equal to 2.

And the last question is just to sketch the graph for the response of this system: $p(D)x$ equals $\cos 2t$, with the initial conditions x of 0 and x dot of 0 equals to 0, basically, rest initial conditions. So why don't you pause the video, take a few minutes, and work through this problem.

Welcome back. So first what is the natural frequency of this system? So let's just rewrite our system here. This is the left-hand side. So basically, this just gives us an x dot dot plus $4x$ on the left-hand side. So the system that we're solving is simply x dot dot plus $4x$ equals $F_0 \cos(\omega t)$.

So the first question asks us for the natural frequency of this system. The natural frequency of this system can be found regardless of what you have the right-hand side, just by looking at the characteristic polynomial of your equation. The characteristic polynomial here would be s squared plus 4. When this characteristic polynomial is equal to 0, we can solve for s and find what are the natural frequencies of the system, if basically we get complex solutions, which is the case here.

Gives us s square equal minus 4. So s equals plus or minus $i \cdot 2$, or $2i$. So the natural frequency of our system would be ω equals 2, because we only consider frequencies that are positive here.

Second part. Now, we're asked to look at the full system with the forcing on the right-hand side. And using the exponential response formula, find one solution to this system. So here we're talking about a particular solution with the exponential response formula.

So what does the ERF tell us? The ERF, if you recall here, the base of it for this system for

example, is the fact that cosine is the real part of the exponential $i\omega t$. So we can rewrite this whole equation as $x'' + 4x = F_0 e^{i\omega t}$.

And we would get then a particular solution, if I ignore any particular value of ω at this point, which would have the form of the amplitude that we have on the right-hand side at 0, exponential $i\omega t$, which is basically our forcing. Over the characteristic polynomial of the equation, so $s^2 + 4$, evaluated at the frequency here that would appear at the forcing in the exponential form, so with the $i\omega t$.

So here you can see right away that we would have a problem, using this formula, if $i\omega t$ was a pole or basically a zero, to this characteristic polynomial. And so that's why you were asked to be careful with the value of ω equal to 2 or not equal to 2.

So here let's consider ω not equal to 2, so that I can actually write down 1 over $p(i\omega)$, because we know that $p(i^2)$ is equal to 0. So if ω is not equal to 2, we're out of the danger zone. And from this point, we can just basically plug in our values, $i\omega t$, and $p(i\omega)$ would just give us $4 - \omega^2$.

So here again, the $\omega = 2$ danger zone approaches, where we would be dividing by 0 if we didn't take the constraint $\omega \neq 2$. So this is the complex form of this particular solution. But we're dealing with a real-valued problem, so we want to take the real part of this to have the solution to the problem we were given. And so that would just give us $F_0 / (4 - \omega^2) \cos \omega t$.

So now let's take the case $\omega = 2$. OK So what happens? If $\omega = 2$, this formula that you're given fails, and you need to seek for the derivative of the characteristic polynomial. And we basically have to-- $2i = 2$. So what about $p'(2i)$? So $p'(s)$ is simply $2s$. So if we evaluate p' at $2i$, we simply have $4i$, which is not equal to 0.

So at this point we can use the resonant exponential response formula that you saw. Just change my chalk. We're here. We would again, same trick, the cosine is just the real part of the exponential. So we can use this formula. And we have now to introduce a t , $F_0 e^{i\omega t}$, because we're solving here for the complex value equation. And now we can divide by the p' evaluated at $2i$, which is $4i$. And so basically, I can end up with a $-i$ at the numerator.

So to take now the real value solution, we need again to take the real part of z_p . So here now

we have an i , so we need to be careful. We're going to have solution in sine. So let me just write down what know. t^*F_0 over 4. This with the Euler formula would be cosine plus i sine. The i sine would be multiplying this i , the two minus would cancel out. And so we would end up with sine ωt , t^*F_0 over 4. And this would then give us the solution.

And here, note that I actually chose the value ω equals to 2, so we can even be more explicit. For this case, we actually have ω equals to $2t$.

So the last part of the problem was to sketch the solution for the initial conditions x of 0 or \dot{x} of 0 equal to 0, so the rest initial conditions. So here are two ways to proceed. The long way would be to seek the solution to the homogeneous equation without the right-hand side, the forcing cosine, introduced two constants of integration, and then seek these constants of integration on the general solution. And you would find that these two constants of integration would be 0 with these initial conditions.

The other fast way to test your particular solution and verify that it actually does satisfy the initial conditions that you were given, and so you can then right away the solution as being simply sine $2t$. Here you can see that at 0, we would have basically a 0. And then if you do a differentiation, you just need to be careful here, because you have a product function, and you end up also with a 0.

So this actually is our general solution for this particular initial condition, And to sketch this, we can grow-- so here if I just pick F_0 equal to 1, I'm just going to do $t/4$ for the envelopes. At t equals 0, we start with 0. And we know that we're going to have the first extrema at $\pi/4$ and the first zero at $\pi/2$.

And so basically, we end up with something like that. So basically, it's sine of circular frequency 2, and with an envelope prescribed by $t/4$. Or if we had another value of F_0 , it would be $F_0 t/4$. So the oscillation is ongoing as t goes to infinity with an envelope that diverges to infinity. So this is basically a solution that would not be convergent to 0.

So this ends this recitation. And before I finish, I just want to point out that the fact that it diverges is due to the fact that we are forcing this system very close to its natural frequency. And so this is a typical phenomenon that you can associate with a resonance, because we're basically forcing a system close to its natural frequency. So it's having this huge amplification in the response. And that's what these increasing envelopes mean.

So this ends this recitation. And the key here was to realize how to use your exponential response formula, how to move on to use the resonance exponential response formula by testing for the first order derivative. And if that test failed, you would be going to higher orders. And then, given an initial condition, how to basically sketch the function and have a physical understanding of what the resonance means. That ends this recitation.