## General Case

It is actually just as easy to write out the formula for the Fourier series expansion of the steady-periodic solution $x_{\text{sp}}(t)$ to the general second-order LTI DE $p(D)x = f(t)$ with $f(t)$ periodic as it was to work out the previous example - the only difference is that now we use letters instead of numbers. We will choose the letters used for the spring-mass-dashpot system, but clearly the derivation and formulas will work with any three parameters.

For simplicity we will take the case of $f(t)$ even (i.e. cosine series).

**Problem:** Solve $m\ddot{x} + b\dot{x} + kx = f(t)$, for the steady-periodic response $x_{\text{sp}}(t)$, where $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right)$

**Solution**

Characteristic polynomial: $p(s) = ms^2 + bs + k$.

Solving for the component pieces:

$m\ddot{x}_n + b\dot{x}_n + kx_n = \cos\left(n\frac{\pi}{L}t\right)$

For $n = 0$ we get $x_{0,p} = \frac{1}{k}$.

For $n \geq 1$:

Complex replacement: $m\ddot{z}_n + b\dot{z}_n + kz_n = e^{in\frac{\pi}{L}t}$, $x_n = \text{Re}(z_n)$

Exponential Response formula: $z_{n,p}(t) = \frac{e^{in\frac{\pi}{L}t}}{p(in\frac{\pi}{L})}$.

Polar coords: $p(in\frac{\pi}{L}) = \left(k - m(n\frac{\pi}{L})^2\right) + i\,b\,n\,\frac{\pi}{L} = |p(in\frac{\pi}{L})|\,e^{i\phi_n}$,

where $|p(in\frac{\pi}{L})| = \sqrt{\left(k - m(n\frac{\pi}{L})^2\right)^2 + b^2(n\frac{\pi}{L})^2}$ and

$\phi_n = \text{Arg}(p(in\frac{\pi}{L})) = \tan^{-1}\left(\frac{b\,n\,\frac{\pi}{L}}{k - m(n\frac{\pi}{L})^2}\right)$ (phase lag).

Thus, $z_{n,p}(t) = g_n\,e^{i(n\frac{\pi}{L}t - \phi_n)}$, with $g_n = \frac{1}{|p(in\frac{\pi}{L})|}$ (gain).

Taking the real part of $x_{n,p}$ we get $x_{n,p}(t) = g_n \cos(n\frac{\pi}{L}t - \phi_n)$.

Now using superposition and putting back in the coefficients $a_n$ we get:

$$x_{\text{sp}}(t) = \frac{a_0}{2}\,x_{0,p} + \sum_{n=1}^{\infty} a_n\,x_{n,p}(t) = \frac{a_0}{2k} + \sum_{n=1}^{\infty} g_n\,a_n\,\cos(n\frac{\pi}{L}t - \phi_n)$$

This is the general formula for the steady periodic response of a second-order LTI DE to an even periodic driver $f(t)$