Repeated Eigenvalues

1. Repeated Eignevalues

Again, we start with the real 2×2 system

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{1}$$

We say an eigenvalue λ_1 of A is **repeated** if it is a multiple root of the characteristic equation of A; in our case, as this is a quadratic equation, the only possible case is when λ_1 is a double real root.

We need to find two linearly independent solutions to the system (1). We can get one solution in the usual way. Let $\mathbf{v}_1$ be an eigenvector corresponding to λ_1. This is found by solving the system

$$(A - \lambda_1 I)\,\mathbf{a} = \mathbf{0}. \tag{2}$$

This gives the solution $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1$ to the system (1). Our problem is to find a second solution. To do this we have to distinguish two cases, called complete and defective. The first one is easier, especially in the 2×2 case.

A. The complete case.
Still assuming λ_1 is a real double root of the characteristic equation of A, we say λ_1 is a **complete** eigenvalue if there are two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to λ_1; i.e., if these two vectors are two linearly independent solutions to the system (2).

In the 2×2 case, this only occurs when A is a *scalar matrix* that is, when $A = \lambda_1 I$. In this case, $A - \lambda_1 I = \mathbf{0}$, and every vector is an eigenvector. It is easy to find two independent solutions; the usual choices are

$$e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e^{\lambda_1 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So the general solution is

$$c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_1 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\lambda_1 t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Of course, we could choose any other pair of independent eigenvectors to generate the solutions, e.g.,

$$e^{\lambda_1 t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad \text{and} \quad e^{\lambda_1 t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Remark. For $n = 3$ and above the situation is more complicated. We will not discuss it here. The interested reader can consult, for instance, the textbook by Edwards and Penney.

B. The defective case.
If the eigenvalue λ is a double root of the characteristic equation, but the system (2) has only one non-zero solution $\mathbf{v}_1$ (up to constant multiples), then the eigenvalue is said to be **incomplete** or **defective** and $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1$ is the unique normal mode. However, a second order system needs two independent solutions. Our experience with repeated roots in second order ODE's suggests we try multiplying our normal solution by t. It turns out this doesn't quite work, but it can be fixed as follows: a second independent solution is given by

$$\mathbf{x}_2 = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{v}_2). \tag{3}$$

where $\mathbf{v}_2$ is any vector satisfying

$$(A - \lambda_1 I)\,\mathbf{v}_2 = \mathbf{v}_1\,.$$

(You can easily, if tediously, check by substitution that this does give a solution. You need to remember that $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$.)

Fact. The equation for $\mathbf{v}_2$ is guaranteed to have a solution, provided that the eigenvalue λ_1 really is defective. When solving for $\mathbf{v}_2 = (b_1, b_2)^T$, try setting $b_1 = 0$, and solving for b_2. If that does not work, try setting $b_2 = 0$ and solving for b_1.

Remarks 1. Some people do not bother with (3). When they encounter the defective case (at least when $n = 2$), they give up on eigenvalues, and simply solve the original system (1) by elimination.

2. Although we will not go into it in this course, there is a well developed theory of defective matrices which gives insight into where this formula comes from. You will learn about all this when you study linear algebra.

We will now do a worked example.

2. Worked example: Defective Repeated Eigenvalue

Problem. Solve $\dot{\mathbf{u}} = A\mathbf{u}$, where $A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$.

Comments are given in italics.

Solution.
Step 0. Write down $A - \lambda I$: $\quad A - \lambda I = \begin{pmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$.

Step 1. Find the characteristic equation of A:
$\text{tr}(A) = -2 + 0 = -2, \quad \det(A) = -2 \times 0 - 1 \times (-1) = 1$. Thus,

$$p_A(\lambda) \;=\; \det(A - \lambda I) \;=\; \lambda^2 - \text{tr}(A)\lambda + \det(A) \;=\; \lambda^2 + 2\lambda + 1 \;=\; 0.$$

Step 2. Find the eigenvalues of A.
The characteristic polynomial factors: $p_A(\lambda) = (\lambda + 1)^2$. This has a repeated root, $\lambda_1 = -1$.

As the matrix A is not the identity matrix, we must be in the defective repeated root case.

Step 3. Find an eigenvector.
This is vector $\mathbf{v}_1 = (a_1, a_2)^T$ that must satisfy:

$$\begin{aligned} (A + I)\mathbf{v}_1 = 0 \qquad &\Leftrightarrow \qquad \begin{pmatrix} -2+1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \qquad \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Check: this gives two identical equations, which is good.
The equation is $-a_1 + a_2 = 0$. Setting $a_1 = 1$ gives $a_2 = 1$. Thus, one eigenvector for λ_1 is $\mathbf{v}_1 = (1, 1)^T$. All other eigenvectors for λ_1 are multiples of this.

Step 4. Find $\mathbf{v}_2$: This vector $\mathbf{v}_2 = (b_1, b_2)^T$ must satisfy

$$(A - \lambda_1 I)\,\mathbf{v}_2 \;=\; \mathbf{v}_1 \quad \Leftrightarrow \quad \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad -b_1 + b_2 = 1.$$

Setting $b_1 = 0$ gives $b_2 = 1$; so $\mathbf{v}_2 = (0, 1)^T$ is suitable.

Step 5. General solution.
The general solution is

$$\mathbf{u}(t) \;=\; c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 (t e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = e^{-t} \left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1+t \end{pmatrix} \right).$$