Class Exam 1 Review Problems –solutions, 18.05, Spring 2022

There are certainly too many problems here to do in class. Pick and choose the ones that will be most helpful to you. The actual test will be much much shorter.

1 Normal probability table

Problem 1. (Table of normal probabilities)
Use the table of standard normal probabilities to compute the following. (Z is the standard normal.)

(a) *(i)* $P(Z \leq 1.5)$ *(ii)* $P(-1.5 < Z < 1.5)$ $P(Z > -0.75)$.

(b) *Suppose* $X \sim N(2, (0.5)^2)$. *Find (i)* $P(X \leq 2)$ *(ii)* $P(1 < X \leq 1.75)$.

Solution: (a) (i) 0.9332 (ii) 0.9332 - 0.0668 = 0.8664

(iii) By symmetry $= P(Z < 0.75) = 0.7734$. (Or we could have used $1 - P(Z > -0.75)$.)

(b) (i) Since 2 is the mean of the normal distribution, $P(X \leq 2) = 0.5$.

(ii) Standardizing,

$$P(1 < X \leq 1.75) = P\left(\frac{1-2}{0.5} < Z \leq \frac{1.75-2}{0.5}\right) = P(-2 < Z < -0.5) = 0.3085\text{–}0.0228 = \boxed{0.2857}.$$

2 Counting and Probability Problems

Problem 2. ***(a)*** *How many ways can you arrange the letters in the word STATISTICS? (e.g. SSSTTTIIAC counts a one arrangement.)*

(b) *If all arrangements are equally likely, what is the probabilitiy the two 'i's are next to each other.*

Solution: (a) Create an arrangement in stages and count the number of possibilities at each stage:

Stage 1: Choose three of the 10 slots to put the S's: $\binom{10}{3}$

Stage 2: Choose three of the remaining 7 slots to put the T's: $\binom{7}{3}$

Stage 3: Choose two of the remaining 4 slots to put the I's: $\binom{4}{2}$

Stage 4: Choose one of the remaining 2 slots to put the A: $\binom{2}{1}$

Stage 5: Use the last slot for the C: $\binom{1}{1}$

Number of arrangements:

$$\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 50400.$$

(b) The are $\binom{10}{2} = 45$ equally likely ways to place the two I's.

There are 9 ways to place them next to each other, i.e. in slots 1 and 2, slots 2 and 3, ..., slots 9 and 10.

So the probability the I's are adjacent is $9/45 = 0.2$.

3 Conditional Probability and Bayes' Theorem Problems

Problem 3. *Corrupted by their power, the judges running the popular game show* America's Next Top Mathematician *have been taking bribes from many of the contestants. Each episode, a given contestant is either allowed to stay on the show or is kicked off.*

If the contestant has been bribing the judges they will be allowed to stay with probability 1. If the contestant has not been bribing the judges, they will be allowed to stay with probability 1/3.

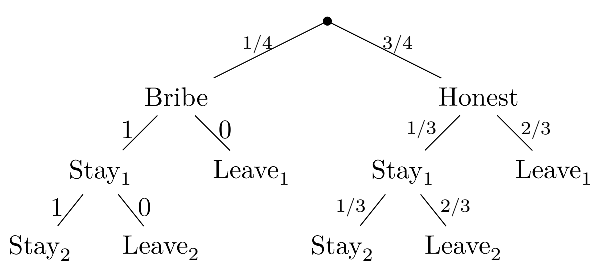
Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds, i.e., if a contestant bribes them in the first round, they bribe them in the second round too (and vice versa).

(a) *If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they were bribing the judges?*

(b) *If you pick a random contestant, what is the probability that they are allowed to stay during both of the first two episodes?*

(c) *If you pick random contestant who was allowed to stay during the first episode, what is the probability that they get kicked off during the second episode?*

Solution: The following tree shows the setting. Stay_1 means the contestant was allowed to stay during the first episode and stay_2 means the they were allowed to stay during the second.



Let's name the relevant events:
B = the contestant is bribing the judges
H = the contestant is honest (not bribing the judges)
S_1 = the contestant was allowed to stay during the first episode
S_2 = the contestant was allowed to stay during the second episode
L_1 = the contestant was asked to leave during the first episode
L_2 = the contestant was asked to leave during the second episode

(a) We first compute $P(S_1)$ using the law of total probability.

$$P(S_1) = P(S_1|B)P(B) + P(S_1|H)P(H) = 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{2}.$$

We therefore have (by Bayes' rule) $P(B|S_1) = P(S_1|B)\dfrac{P(B)}{P(S_1)} = 1 \cdot \dfrac{1/4}{1/2} = \dfrac{1}{2}.$

(b) Using the tree we have the total probability of S_2 is

$$P(S_2) = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}$$

(c) We want to compute $P(L_2|S_1) = \dfrac{P(L_2 \cap S_1)}{P(S_1)}.$

From the calculation we did in part (a), $P(S_1) = 1/2$. For the numerator, we have (see the tree)

$$P(L_2 \cap S_1) = P(L_2 \cap S_1|B)P(B) + P(L_2 \cap S_1|H)P(H) = 0 \cdot \frac{1}{4} + \frac{2}{9} \cdot \frac{3}{4} = \frac{1}{6}$$

Therefore $\boxed{P(L_2|S_1) = \dfrac{1/6}{1/2} = \dfrac{1}{3}.}$

4 Independence Problems

Problem 4. *You roll a twenty-sided die. Determine whether the following pairs of events are independent.*

(a) *'You roll an even number' and 'You roll a number less than or equal to 10'.*

(b) *'You roll an even number' and 'You roll a prime number'.*

Solution: E = even numbered = $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.

L = roll ≤ 10 = $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

B = roll is prime = $\{2, 3, 5, 7, 11, 13, 17, 19\}$ (We use B because P is not a good choice.)

(a) $P(E) = 10/20$, $P(E|L) = 5/10$. These are the same, so the events are independent.

(b) $P(E) = 10/20$. $P(E|B) = 1/8$. These are not the same so the events are not independent.

5 Expectation and Variance Problems

Problem 5. *The random variable X takes values -1, 0, 1 with probabilities 1/8, 2/8, 5/8 respectively.*

(a) *Compute $E[X]$.*

(b) *Give the pmf of $Y = X^2$ and use it to compute $E[Y]$.*

(c) *Instead, compute $E[X^2]$ directly from an extended table.*

(d) *Compute* Var(X).

(a) Solution: We have

X values:	-1	0	1
prob:	1/8	2/8	5/8
X^2	1	0	1

So, $\boxed{E[X] = -1/8 + 5/8 = 1/2.}$

(b) Solution:

Y values:	0	1
prob:	2/8	6/8

$\Rightarrow$ $\boxed{E[Y] = 6/8 = 3/4.}$

(c) Solution: The change of variables formula just says to use the bottom row of the table in part (a): $E[X^2] = 1 \cdot (1/8) + 0 \cdot (2/8) + 1 \cdot (5/8) = 3/4$ (same as part (b)).

(d) Solution: $\boxed{\text{Var}(X) = E[X^2] - E[X]^2 = 3/4 - 1/4 = 1/2.}$

Problem 6. *Suppose 100 people all toss a hat into a box and then proceed to randomly pick out of a hat. What is the expected number of people to get their own hat back.*

Hint: express the number of people who get their own hat as a sum of random variables whose expected value is easy to compute.

Solution: Let X be the number of people who get their own hat.

Following the hint: let X_j represent whether person j gets their own hat. That is, $X_j = 1$ if person j gets their hat and 0 if not.

We have, $X = \sum_{j=1}^{100} X_j$, so $E[X] = \sum_{j=1}^{100} E[X_j]$.

Since person j is equally likely to get any hat, we have $P(X_j = 1) = 1/100$. Thus, $X_j \sim$ Bernoulli$(1/100)$ $\Rightarrow$ $E[X_j] = 1/100$ $\Rightarrow$ $\boxed{E[X] = 1.}$

6 Probability Mass Functions, Probability Density Functions and Cumulative Distribution Functions Problems

Problem 7. ***(a)*** *Suppose that X has probability density function $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Compute the cdf, $F_X(x)$.*

(b) *If $Y = X^2$, compute the pdf and cdf of Y.*

(a) Solution: We have cdf of X,

$$F_X(x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}.$$

Now for $y \geq 0$, we have

(b) Solution:

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = 1 - e^{-\lambda \sqrt{y}}.$$

Differentiating $F_Y(y)$ with respect to y, we have

$$f_Y(y) = \frac{\lambda}{2} y^{-\frac{1}{2}} e^{-\lambda\sqrt{y}}.$$

Problem 8. *Suppose you roll a fair 6-sided die 100 times (independently), and you get \$3 every time you roll a 6.*

Let X_1 be the number of dollars you win on rolls 1 through 25.
Let X_2 be the number of dollars you win on rolls 26 through 50.
Let X_3 be the number of dollars you win on rolls 51 through 75.
Let X_4 be the number of dollars you win on rolls 76 throught 100.

Let $X = X_1 + X_2 + X_3 + X_4$ be the total number of dollars you win over all 100 rolls.

(a) *What is the probability mass function of X?*

(b) *What is the expectation and variance of X?*

(c) Let $Y = 4X_1$. *(So instead of rolling 100 times, you just roll 25 times and multiply your winnings by 4.)*

(i) What are the expectation and variance of Y?

(ii) How do the expectation and variance of Y compare to those of X? (That is, are they bigger, smaller, or equal?) Explain (briefly) why this makes sense.

Solution: (a) There are a number of ways to present this.

Let T be the total number of times you roll a 6 in the 100 rolls. We know $T \sim \text{Binomial}(100, 1/6)$. Since you win \$3 every time you roll a 6, we have $X = 3T$. So, we can write

$$P(X = 3k) = \binom{100}{k}\left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{100-k}, \quad \text{for } k = 0,\ 1,\ 2,\ \ldots,\ 100.$$

Alternatively we could write

$$P(X = x) = \binom{100}{x/3}\left(\frac{1}{6}\right)^{x/3} \left(\frac{5}{6}\right)^{100-x/3}, \quad \text{for } x = 0,\ 3,\ 6,\ \ldots,\ 300.$$

(b) $E[X] = E[3T] = 3E[T] = 3 \cdot 100 \cdot \frac{1}{6} = 50$,

$\text{Var}(X) = \text{Var}(3T) = 9\text{Var}(T) = 9 \cdot 100 \cdot \frac{1}{6} \cdot \frac{5}{6} = 125$.

(c) (i) Let T_1 be the total number of times you roll a 6 in the first 25 rolls. So, $X_1 = 3T_1$ and $Y = 12T_1$.

Now, $T_1 \sim \text{Binomial}(25, 1/6)$, so

$$E[Y] = 12E[T_1] = 12 \cdot 25 \cdot 16 = 50.$$

and

$$\text{Var}(Y) = 144\text{Var}(T_1) = 144 \cdot 25 \cdot \frac{1}{6} \cdot \frac{5}{6} = 500.$$

(ii) The expectations are the same by linearity because X and Y are the both

$3 \times 100 \times$ a Bernoulli(1/6) random variable.

For the variance, $\text{Var}(X) = 4\text{Var}(X_1)$ because X is the sum of 4 *independent* variables all identical to X_1. However $\text{Var}(Y) = \text{Var}(4X_1) = 16\text{Var}(X_1)$. So, the variance of Y is 4 times that of X. This should make some intuitive sense because X is built out of more independent trials than X_1.

Another way of thinking about it is that the difference between Y and its expectation is four times the difference between X_1 and its expectation. However, the difference between X and its expectation is the sum of such a difference for X_1, X_2, X_3, and X_4. It's probably the case that some of these deviations are positive and some are negative, so the absolute value of this difference for the sum is probably less than four times the absolute value of this difference for one of the variables, i.e. the deviations are likely to cancel to some extent.

7 Joint Probability, Covariance, Correlation Problems

Problem 9. Covariance and Independence
Let X be a random variable that takes values -2, -1, 0, 1, 2; each with probability 1/5. Let $Y = X^2$.

(a) *Fill out the following table giving the joint frequency function for X and Y. Be sure to include the marginal probabilities.*

X / Y	-2	-1	0	1	2	total
0						
1						
4						
total						

Solution:

X / Y	-2	-1	0	1	2	
0	0	0	1/5	0	0	1/5
1	0	1/5	0	1/5	0	2/5
4	1/5	0	0	0	1/5	2/5
	1/5	1/5	1/5	1/5	1/5	1

Each column has only one nonzero value. For example, when $X = -2$ then $Y = 4$, so in the $X = -2$ column, only $P(X = -2, Y = 4)$ is not 0.

(b) *Find $E[X]$ and $E[Y]$.*

Solution: Using the marginal distributions: $E[X] = \frac{1}{5}(-2 - 1 + 0 + 1 + 2) = 0$.

$$E[Y] = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 4 \cdot \frac{2}{5} = 2.$$

(c) *Show X and Y are not independent.*

Solution: We show the probabilities don't multiply:

$P(X = -2,\ Y = 0) = 0 \neq P(X = -2) \cdot P(Y = 0) = 1/25.$

Since these are not equal X and Y are not independent. (It is obvious that X^2 is not independent of X.)

(d) *Show* $\text{Cov}(X, Y) = 0$.
This is an example of uncorrelated but non-independent random variables. The reason this can happen is that correlation only measures the linear dependence between the two variables. In this case, X and Y are not at all linearly related.

Solution: Using the table from part (a) and the means computed in part (d) we get:

$$\begin{aligned}\text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\ &= \frac{1}{5}(-2)(4) + \frac{1}{5}(-1)(1) + \frac{1}{5}(0)(0) + \frac{1}{5}(1)(1) + \frac{1}{5}(2)(4) \\ &= 0.\end{aligned}$$

Problem 10. Continuous Joint Distributions
Suppose X and Y are continuous random variables with joint density function $f(x, y) = c(x + 2y)$ on the rectangle $[0, 1] \times [0, 2]$.

When doing this problem in class: Only compute the integrals in parts (a) and (b). For the others, just give the integrals in a form like $\int_0^1 \int_0^2 x f(x, y)\, dy\, dx$, but don't compute them.

(a) *Find the value of c.*

Solution: We need the total probability to be 1. So

$$\int_0^1 \int_0^2 c(x + 2y)\, dy\, dx = 1.$$

Inner integral: $c(xy + y^2)\Big|_0^2 = c(2x + 4)$.

Outer integral: $\left[cx^2 + 4x\right]_0^1 = 5c$.

So $\boxed{c = 1/5.}$

(b) *Let $F(x, y)$ be the joint CDF. Compute $F(x, y)$. Compute $F(1, 2)$.*

Solution: $F(a, b) = P(X \le a, Y \le b) = \int_0^a \int_0^b f(x, y)\, dx\, dy = \frac{1}{5} \int_0^a \int_0^b (x + 2y)\, dy\, dx.$

Inner integral: $\frac{1}{5}\left[xy + y^2\right]_0^b = \frac{1}{5}(xb + b^2)$.

Outer integral: $\frac{1}{5}\left[\frac{bx^2}{2} + b^2 x\right]_0^a = \frac{1}{5}\left(\frac{a^2 b}{2} + ab^2\right)$.

So, changing back to x, y: $\boxed{F(x, y) = \frac{1}{5}\left(\frac{x^2 y}{2} + xy^2\right)}$ and $\boxed{F(1, 2) = 1.}$

(c) *Compute the marginal densities for X and Y.*

Solution: $f_X(x) = \int_0^2 f(x, y)\, dy = \frac{1}{5} \int_0^2 (x + 2y)\, dy = \frac{1}{5}\left[xy + y^2\right]_0^2 = \boxed{\frac{1}{5}(2x + 4).}$

Similarly, $\boxed{f_Y(y) = \int_0^1 f(x, y)\, dx = \frac{1}{5}\left(\frac{1}{2} + 2y\right).}$

(d) *Are X and Y independent?*

Solution: To see if they are independent we check if the joint density is the product of the marginal densities.

$$f(x,y) = x+y,\ f_X(x)\cdot f_Y(y) = \frac{1}{25}(2x+4)(1/2+2y).$$

Since these are not equal, $\boxed{X \text{ and } Y \text{ are not independent.}}$

(e) *Compute $E[X]$, $E[Y]$, $E[X^2+Y^2]$, $\mathrm{Cov}(X,Y)$, $Cor(X,Y)$.*

Solution:

$$E[X] = \int_0^1\int_0^2 xf(x,y)\,dy\,dx = \frac{1}{5}\int_0^1\int_0^2 x(x+2y)\,dy\,dx.$$

The integration is similar to all the others in this problem: $\boxed{E[X] = \frac{8}{15} \approx 0.533.}$

(Or, using (b), $E[X] = \int_0^1 xf_X(x)\,dx = \frac{1}{5}\int_0^1 x(2x+4)\,dx = 8/15.$)

Similarly, $E[Y] = \int_0^1\int_0^2 yf(x,y)\,dy\,dx = \boxed{\frac{19}{15} \approx 1.267.}$.

For the remaining integrals, we won't show any computation. They all look similar.

$$E[X^2+Y^2] = \int_0^1\int_0^2 (x^2+y^2)f(x,y)\,dy\,dx = \frac{1}{5}\int_0^1\int_0^2 (x^2+y^2)(x+2y)\,dy\,dx = \boxed{\frac{67}{30} \approx 2.333.}$$

For $\mathrm{Cov}(X,Y)$ we use the formula $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$.

$$E[XY] = \int_0^1\int_0^2 xyf(x,y)\,dy\,dx = \frac{1}{5}\int_0^1\int_0^2 xy(x+2y)\,dy\,dx = \boxed{\frac{2}{3} \approx 0.667.}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = \boxed{\frac{2}{3} - \frac{8}{15}\cdot\frac{19}{15} \approx -0.00889.}$$

To find the covariance, we need the variances of X and Y. For this, we first compute $E[X^2]$, $E[Y^2]$.

$$E[X^2] = \int_0^1\int_0^2 x^2f(x,y)\,dy\,dx = 11/30 \ \text{ and } \ E[Y^2] = \int_0^1\int_0^2 y^2f(x,y),dy\,dx = 28/15.$$

So, $\mathrm{Var}(X) = 11/30 - (8/15)^2 \approx 0.0822$ and $\mathrm{Var}(Y) = 28/15 - (19/15)^2 \approx 0.262$.

Thus, $\sigma_X \approx \sqrt{0.0822}$, $\sigma_Y \approx \sqrt{0.262}$ and $\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} \approx \boxed{-0.0606.}$.

8 Law of Large Numbers, Central Limit Theorem Problems

Problem 11. *Suppose $X_1,\ldots,X_{100}$ are i.i.d. with mean 1/5 and variance 1/9. Use the central limit theorem to estimate $P(\sum X_i < 30)$.*

Solution: Standardize:

$$\begin{aligned} P\left(\sum_i X_i < 30\right) &= P\left(\frac{\sum X_i - \mu}{\sqrt{n}\,\sigma} < \frac{30 - n\mu}{\sqrt{n}\,\sigma}\right) \\ &\approx P\left(Z < \frac{30-20}{10/3}\right) \quad \text{(by the central limit theorem)} \\ &= P(Z < 3) \\ &= 0.9987 \quad \text{(from the table of normal probabilities)} \end{aligned}$$

Problem 12. (**More Central Limit Theorem**)
The average IQ in a population is 100 with standard deviation 15 (by definition, IQ is normalized so this is the case). What is the probability that a randomly selected group of 100 people has an average IQ above 115?

Solution: Let X_j be the IQ of a randomly selected person. We are given $E[X_j] = 100$ and $\sigma_{X_j} = 15$.
Let $\overline{X}$ be the average of the IQ's of 100 randomly selected people. Then we know

$$E[\overline{X}] = 100 \quad \text{and} \quad \sigma_{\overline{X}} = 15/\sqrt{100} = 1.5.$$

The problem asks for $P(\overline{X} > 115)$. Standardizing we get $P(\overline{X} > 115) \approx P(Z > 10)$. $\boxed{\text{This is effectively 0.}}$

9 More problems

Problem 13. (**Arithmetic Puzzle**)
The joint and marginal pmf's of X and Y are partly given in the following table.

$X \backslash Y$	1	2	3	
1	1/6	0	...	1/3
2	...	1/4	...	1/3
3	...	...	1/4	...
	1/6	1/3	...	1

(a) *Complete the table.*

(b) *Are X and Y independent?*

Solution: **(a)** The marginal probabilities have to add up to 1, so the two missing marginal probabilities can be computed: $P(X = 3) = 1/3$, $P(Y = 3) = 1/2$. Now each row and column has to add up to its respective margin. For example, $1/6 + 0 + P(X = 1, Y = 3) = 1/3$, so $P(X = 1, Y = 3) = 1/6$. Here is the completed table.

$X \backslash Y$	1	2	3	
1	1/6	0	1/6	1/3
2	0	1/4	1/12	1/3
3	0	1/12	1/4	1/3
	1/6	1/3	1/2	1

(b) No, X and Y are not independent.

For example, $P(X = 2, Y = 1) = 0 \neq P(X = 2) \cdot P(Y = 1)$.

Problem 14. *Compute the expectation and variance of a Bernoulli(p) random variable.*

Solution: Make a table:

X:	0	1
prob:	(1-p)	p
X^2	0	1.

From the table, $\boxed{E[X] = 0 \cdot (1 - p) + 1 \cdot p = p.}$

Since X and X^2 have the same table $E[X^2] = E[X] = p$.

Therefore, $\boxed{\text{Var}(X) = p - p^2 = p(1 - p).}$