Exam 1 Practice Questions II, 18.05, Spring 2022

Notes.

Not every possible question be covered in 11 problems. Look at the other review problems as well as the readings, psets and class problems.

Even the first 11 problems are much longer than the actual test will be,

Problem 1. A full house in poker is a hand where three cards share one rank and two cards share another rank. How many ways are there to get a full-house? What is the probability of getting a full-house?

Problem 2. Let C and D be two events with $P(C) = 0.25$, $P(D) = 0.45$, and $P(C \cap D) = 0.1$. What is $P(C^c \cap D)$?

Problem 3. Corrupted by their power, the judges running the popular game show *America's Next Top Mathematician* have been taking bribes from many of the contestants. Each episode, a given contestant is either allowed to stay on the show or is kicked off.

If the contestant has been bribing the judges they will be allowed to stay with probability 1. If the contestant has not been bribing the judges, they will be allowed to stay with probability 1/3.

Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds, i.e., if a contestant bribes them in the first round, they bribe them in the second round too (and vice versa).

(a) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they were bribing the judges?

(b) If you pick a random contestant, what is the probability that they are allowed to stay during both of the first two episodes?

(c) If you pick random contestant who was allowed to stay during the first episode, what is the probability that they get kicked off during the second episode?

Problem 4. Suppose now that events A, B and C are *mutually independent* with

$$P(A) = 0.3, \quad P(B) = 0.4, \quad P(C) = 0.5.$$

Compute the following: (Hint: Use a Venn diagram)
(i) $P(A \cap B \cap C^c)$ (ii) $P(A \cap B^c \cap C)$ (iii) $P(A^c \cap B \cap C)$

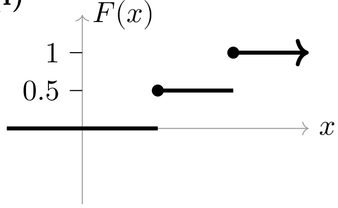
Problem 5. Suppose X is a random variable with $E[X] = 5$ and $\mathrm{Var}(X) = 2$. What is $E[X^2]$?

Problem 6. **(a)** Suppose that X has probability density function $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Compute the cdf, $F_X(x)$.

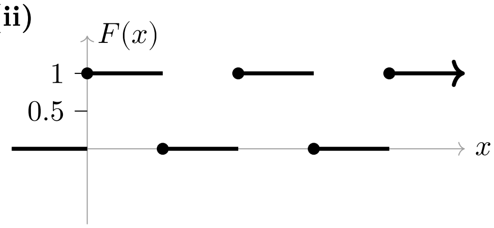
(b) If $Y = X^2$, compute the pdf and cdf of Y.

Problem 7. For each of the following say whether it can be the graph of a cdf. If it can be, say whether the variable is discrete or continuous.

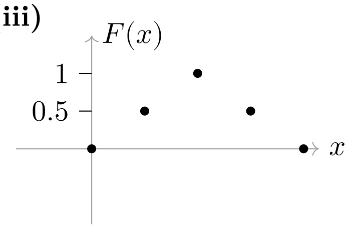
(i)



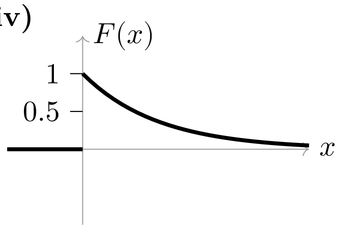
(ii)



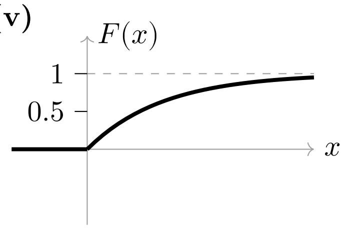
(iii)



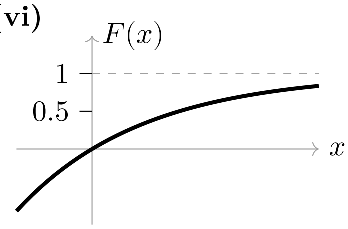
(iv)



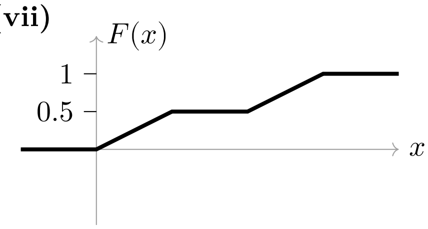
(v)



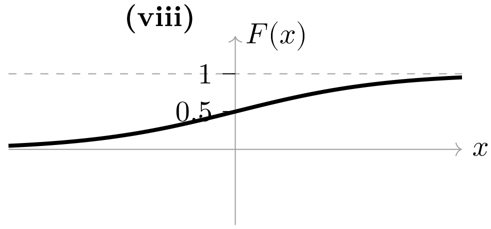
(vi)



(vii)



(viii)



Problem 8. Compute the median for the exponential distribution with parameter λ.

Problem 9. (**Another Arithmetic Puzzle**)
Let X and Y be two independent Bernoulli(0.5) random variables. Define S and T by:

$$S = X + Y \quad \text{and} \quad T = X - Y.$$

(a) Find the joint and marginal pmf's for S and T.

(b) Are S and T independent.

Problem 10. Let X and Y be two random variables and let r, s, t, and u be real numbers.
(a) Show that $\text{Cov}(X + s, Y + u) = \text{Cov}(X, Y)$.

(b) Show that $\text{Cov}(rX, tY) = rt\text{Cov}(X, Y)$.

(c) Show that $\text{Cov}(rX + s, tY + u) = rt\text{Cov}(X, Y)$.

Problem 11. (**More Central Limit Theorem**)
The average IQ in a population is 100 with standard deviation 15 (by definition, IQ is normalized so this is the case). What is the probability that a randomly selected group of 100 people has an average IQ above 115?

More problems

Problem 12. 20 politicians are having a tea party, 6 Democrats and 14 Republicans. To prepare, they need to choose:

3 people to set the table, 2 people to boil the water, 6 people to make the scones.

Each person can only do 1 task. (Note that this doesn't add up to 20. The rest of the people don't help.)

(a) In how many different ways can they choose which people perform these tasks?

(b) Suppose that the Democrats all hate tea. If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

(c) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 9 Republicans and 1 Democrat?

Problem 13. More cards! Suppose you want to divide a 52 card deck into four hands with 13 cards each. What is the probability that each hand has a king?

Problem 14. There is a screening test for prostate cancer that looks at the level of PSA (prostate-specific antigen) in the blood. There are a number of reasons besides prostate cancer that a man can have elevated PSA levels. In addition, many types of prostate cancer develop so slowly that that they are never a problem. Unfortunately there is currently no test to distinguish the different types and using the test is controversial because it is hard to quantify the accuracy rates and the harm done by false positives.

For this problem we'll call a positive test a true positive if it catches a dangerous type of prostate cancer. We'll assume the following numbers:

Rate of prostate cancer among men over 50 = 0.0005
True positive rate for the test = 0.9
False positive rate for the test = 0.01

Let T be the event a man has a positive test and let D be the event a man has a dangerous type of the disease. Find $P(D|T)$ and $P(D|T^c)$.

Problem 15. Let X be a discrete random variable with pmf p given by:

x	-2	-1	0	1	2
$p(x)$	1/15	2/15	3/15	4/15	5/15

(a) Let $Y = X^2$. Find the pmf of Y.

(b) Find the value the cdf of X at -3/2, 3/4, 7/8, 1, 1.5, 5.

(c) Find the value the cdf of Y at -3/2, 3/4, 7/8, 1, 1.5, 5.

Problem 16. Suppose I play a gambling game with even odds. So, I can wager b dollars and I either win or lose b dollars with probability $p = 0.5$.

I employ the following strategy to try to guarantee that I win some money.

I bet \$1; if I lose, I double my bet to \$2, if I lose I double my bet again. I continue until I win. Eventually I'm sure to win a bet and net \$1 (run through the first few rounds and you'll see why this is the net).

If this really worked casinos would be out of business. Our goal in this problem is to understand the flaw in the strategy.

(a) Let X be the amount of money bet on the last game (the one I win). X takes values 1, 2, 4, 8, ... Determine the probability mass function for X. That is, find $p(2^k)$, where k is in {0, 1, 2, ...}.

(b) Compute $E[X]$.

(c) Use your answer in part (b) to explain why the stategy is a bad one.

Problem 17. Normal Distribution: Throughout these problems, let ϕ and Φ be the pdf and cdf, respectively, of the standard normal distribution Suppose Z is a standard normal random variable and let $X = 3Z + 1$.

(a) Express $P(X \leq x)$ in terms of Φ

(b) Differentiate the expression from (a) with respect to x to get the pdf of X, $f(x)$. Remember that $\Phi'(z) = \phi(z)$ and don't forget the chain rule

(c) Find $P(-1 \leq X \leq 1)$

(d) Recall that the probability that Z is within one standard deviation of its mean is approximately 68%. What is the probability that X is within one standard deviation of its mean?

Problem 18. Toss a fair coin 3 times. Let X = the number of heads on the first toss, Y the total number of heads on the last two tosses, and F the number of heads on the first two tosses.

(a) Give the joint probability table for X and Y. Compute $\text{Cov}(X, Y)$.

(b) Give the joint probability table for X and F. Compute $\text{Cov}(X, F)$.