Exercises on the geometry of linear equations

Problem 1.1: (1.3 #4. *Introduction to Linear Algebra:* Strang) Find a combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3$ that gives the zero vector:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent)(dependent).

The three vectors lie in a ___________. The matrix W with those columns is *not invertible*.

Solution: We might observe that $\mathbf{w}_1 + \mathbf{w}_3 - 2\mathbf{w}_2 = 0$, or we might simultaneously solve the system of equations:

$$\begin{aligned} 1x_1 + 4x_2 + 7x_3 &= 0 \\ 2x_1 + 5x_2 + 8x_3 &= 0 \\ 3x_1 + 6x_2 + 9x_3 &= 0 \end{aligned}$$

Subtracting twice equation 1 from equation 2 gives us $-3x_2 - 6x_3 = 0$. Subtracting thrice equation 1 from equation 3 gives us $-6x_2 - 12x_3 = 0$, which is equivalent to the previous equation and so leads us to suspect that the vectors are dependent. At this point we might guess $x_2 = -2$ and $x_3 = 1$ which would lead us to the answer we observed above:

$$x_1 = 1,\ x_2 = -2,\ x_3 = 1 \text{ and } \mathbf{w}_1 - 2\mathbf{w}_2 + \mathbf{w}_3 = 0.$$

Those vectors are **dependent** because there is a combination of the vectors that gives the zero vector.

The three vectors lie in a **plane.**

Problem 1.2: Multiply: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Solution: $\begin{bmatrix} 1 \cdot 3 + 2 \cdot (-2) + 0 \cdot 1 \\ 6 + 0 + 3 \\ 12 - 2 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$.

Problem 1.3: True or false: A 3 by 2 matrix A times a 2 by 3 matrix B equals a 3 by 3 matrix AB. If this is false, write a similar sentence which is correct.

Solution: The statement is true. In order to multiply two matrices, the number of columns of A must equal the number of rows of B. The product AB will have the same number of rows as the first matrix and the same number of columns as the second:

$$A(m \text{ by } n) \text{ times } B(n \text{ by } p) \text{ equals } AB(m \text{ by } p).$$