18.085 Computational Science and Engineering I
Fall 2008

18.085 Quiz 3   December 7, 2007   Professor Strang

Your PRINTED name is: _____

** <u>NOTE</u> AT NOON A BIG CHEMISTRY CLASS IS COMING !!!

1) **(30 pts.)**   (a) Solve by a Fourier sine series $u(x) = \sum b_k \sin kx$:

$$-u'' + 4\,u(x) = f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases} \quad \text{with} \quad u(-\pi) = u(\pi) = 0.$$

That right side $f(x)$ is the square wave $SW(x)$ on page 318.

(b) What is the decay rate of the coefficients b_k? What is the smoothness of $u(x)$ — which derivative jumps?

We're told that the solution $u(x)$ has the form of a Fourier sine series, $u(x) = \sum_{k=1}^{\infty} b_k \sin kx$. The problem is to find the coefficients b_k.

I think the easiest way to do this[1] is to take the given differential equation and substitute in (1) the Fourier series for $u(x)$ and (2) the Fourier series for the right hand side given on page 318 of the textbook.

$$-\frac{d^2}{dx^2} \sum_{k=1}^{\infty} b_k \sin kx + 4 \sum_{k=1}^{\infty} b_k \sin kx = \frac{4}{\pi} \sum_{k \text{ odd}} \frac{1}{k} \sin kx$$

$$\sum_{k=1}^{\infty} (k^2 + 4) b_k \sin kx = \frac{4}{\pi} \sum_{k \text{ odd}} \frac{1}{k} \sin kx$$

Setting the coefficients of $\sin kx$ on the left and right sides equal to each other gives the following equations for the b_k's.

$$(k^2 + 4) b_k = \left\{ \begin{array}{ll} \frac{4}{\pi k} & k = 1, 3, 5, ... \\ 0 & k = 2, 4, 6, ... \end{array} \right\}$$

So, for odd k,

$$b_k = \frac{4}{\pi k (k^2 + 4)},$$

and, for even k, $b_k = 0$. The solution is

$$u(x) = \sum_{k \text{ odd}} \frac{4}{\pi k (k^2 + 4)} \sin kx.$$

Since the denominators of the coefficients contain a k^3, the decay rate is $1/k^3$. By comparison with the chart on page 321, this means the first derivative $u'(x)$ is continuous but the second derivative $u''(x)$ jumps.

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[1] Technically, you can't take the Fourier transform here because $u(x)$ and $f(x)$ are functions on the interval $[-\pi, \pi]$, not the entire real line $(-\infty, \infty)$.

2) **(30 pts.)**   This problem is about the equation

$$\frac{1}{5} x_{n-1} + \frac{3}{5} x_n + \frac{1}{5} x_{n+1} = y_n \qquad -\infty < n < \infty$$

(a) Suppose the vector $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$ is known. The equation is a non-cyclic convolution $a * x = y$. <u>What is the infinite vector a?</u> Transform the equation into the frequency domain using $X(\omega) = \sum x_n e^{in\omega}$ and $Y(\omega)$ and $A(\omega)$. What is $A(\omega)$ in this problem?

(b) Suppose the vector y is known but the vector x is **not known**. We want to find x. Take two steps:

   1. Give a formula for $X(\omega)$ using known things like $Y(\omega)$ and $\frac{1}{5}, \frac{3}{5}, \frac{1}{5}$, or A.

   2. Does your formula involve any division by zero or is it safe?

   The last step in this deconvolution would recover the Fourier coefficients x_n from your function $X(\omega)$ but this is not on the exam!

The first part asks for the vector a that, when you take the convolution $a * (..., x_{-1}, x_0, x_1, ...)$, gives you the vector y that has each component $y_n = \frac{1}{5}x_{n-1} + \frac{3}{5}x_n + \frac{1}{5}x_{n+1}$. You can check that the vector

$$a = (..., 0, 0, 0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0, 0, 0, ...)$$

will accomplish this. (Here, $a_0 = \frac{3}{5}$, $a_1 = a_{-1} = \frac{1}{5}$, and all the other a_n's are zero.)

I think the easiest way to think about this problem is to think of the vectors x, y, and a (with components x_n, y_n, and a_n for $n = -\infty...\infty$) as the Fourier coefficients of the functions $X(\omega)$, $Y(\omega)$, and $A(\omega)$. In other words:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x_n e^{in\omega}$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y_n e^{in\omega}$$

$$A(\omega) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega}$$

So, substituting the coefficients a_n above into the Fourier series for $A(\omega)$ gives you

$$A(\omega) = \frac{1}{5}e^{-i\omega} + \frac{3}{5} + \frac{1}{5}e^{i\omega}$$

$$= \frac{3}{5} + \frac{2}{5}\cos\omega.$$

Either of the above answers is correct but the second one is easier to visualize.

To transform the problem into the frequency domain, use the first convolution rule in the box on page 356.

$$a * x = y$$

$$A(\omega)X(\omega) = Y(\omega)$$

This directly gives a formula for $X(\omega)$.

$$X(\omega) = \frac{Y(\omega)}{A(\omega)}$$

$$= \frac{Y(\omega)}{\frac{3}{5} + \frac{2}{5}\cos\omega}$$

In general, dividing by some arbitrary $A(\omega)$ could be a problem, but in this case, since $\cos\omega$ ranges from -1 to 1, the denominator can never be zero, so the formula is safe.

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3) **(40 pts.)** This circulant equation $Cd = b$ is a cyclic convolution:

$$Cd = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = b \quad \text{is} \quad c \circledast d = b$$

(a) The eigenvectors of that matrix C are the four columns e_0, e_1, e_2, e_3 of the Fourier matrix F (this F is on page 347). Multiply F times the e's to find the four <u>eigenvalues</u>. Check that their sum is correct.

(b) Write the right side $b = (1, 0, 0, 0)$ as a combination of those four eigenvectors (columns of F). Using the eigenvalues, the solution d is what combination of the four eigenvectors? <u>Find the vector d</u>.

(c) A direct way to solve $c \circledast d = b$ would be to take the 4-point discrete transform of both sides. What are the transforms of b and c in this problem? What is the transform of the solution d? Isn't this just the same method in different words (yes or no).

Thank you for taking 18.085 !

There are two methods for computing the eigenvalues of C. The first, as outlined in the exam, is to simply multiply the columns of the Fourier matrix F by the matrix C and see which multiples we get. This will always work since the columns of the Fourier matrix are always eigenvectors for a circulant matrix. In fact, though, since we already know this, we only need to check how the first entry in each column of F changes. Since that first entry is always a 1, whatever it changes to is the eigenvalue. Thus, to get the eigenvalues of C we can simply multiply F on the left by the first row of C, and that list will be the list of eigenvalues.

This seems quite complicated in words, so first I will carry out the multiplications in full, and then demonstrate the faster technique:

$$
C \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad
C \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} 5 \\ 5i \\ -5 \\ -5i \end{bmatrix}
$$

$$
C \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 5 \\ -5 \end{bmatrix} \qquad
C \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} = \begin{bmatrix} 5 \\ -5i \\ -5 \\ 5i \end{bmatrix}
$$

So the eigenvalues corresponding to e_0, e_1, e_2, and e_3 are 1, 5, 5, and 5 respectively.

In the other method we would have simply multiplied

$$
\begin{bmatrix} 4 & -1 & -1 & -1 \end{bmatrix}
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}
= \begin{bmatrix} 1 & 5 & 5 & 5 \end{bmatrix}
$$

to get the same answer.

The sum of the eigenvalues is 16, which is the trace of C: good.

We can almost "eyeball" the eigenvectors to see what combination b is, but alternatively we

can multiply b by F^{-1}:

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

so that $b = \frac{1}{4}(e_0 + e_1 + e_2 + e_3)$. We know then that

$$d = \frac{1}{4}\left(e_0 + \frac{1}{5}e_1 + \frac{1}{5}e_2 + \frac{1}{5}e_3\right)$$

as can easily be checked by multiplying the right side by C. Plugging in the eigenvectors we see that

$$d = \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

For the last part of the problem we use the formula:

$$F(c \circledast d) = Fc. * Fd$$

Then, taking the transforms we get

$$Fc = F\begin{bmatrix} 4 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$

$$Fb = F\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

So that

$$Fd = \begin{bmatrix} 1 \\ 1/5 \\ 1/5 \\ 1/5 \end{bmatrix}$$

If we wished to find d now, we could multiply by F^{-1} and get the same answer as before. As for the last question, "Isn't this just the same method in different words", either answer was accepted. The method of part b makes it a little clearer to me why the method of part c works, but on the other hand, I can imagine people feeling the other way. The latter method is somehow more general, just because I can't imagine how to program the former method without using the latter.