Partial Differential Equations

3.7 Four Model Examples

The differential equations in Chapter 1 were very ordinary. There were time derivatives d/dt or space derivatives d/dx but not both:

$$\frac{du}{dt} = -Ku \quad \text{or} \quad M\frac{d^2u}{dt^2} + Ku = F(t) \quad \text{or} \quad -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x)\,. \tag{1}$$

A partial differential equation contains two or more derivatives (they have to be partial derivatives like $\partial/\partial x$ and $\partial/\partial y$ and $\partial/\partial t$ so we can tell them apart). The solution $u(x,y)$ or $u(x,t)$ or even $u(x,y,t)$ is a function of those "independent variables" x and y and t.

It is important to distinguish different types of equations, above all the distinction between "boundary value problems" and initial value problems". The time variable t indicates an initial value problem. The first equation in (1) starts from an initial value $u(0)$. The solution $u(\epsilon)$ evolves?? for $t > 0$ by obeying the equation $du/dt = Au$. The second equation needs also an initial value $du/dt(0)$ for the velocity, because the leading term involves d^2u/dt^2. Boundary values were given at endpoints $x = 0$ and $x = 1$. Inside the boundary (in the interior) $u(x)$ solved the equation, with just enough freedom (two arbitrary constants) to satisfy the two boundary conditions. All good. The third equation in (1) described a **steady state** $u(x)$.

For partial differential equations, start with initial value problems. We will focus on three examples. They involve first or second order derivatives in t and in x and $u(x,t)$ is a scalar. The names of the equations are important too:

One way wave equation $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \tag{2}$$

Heat equation, diffusion equation $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Wave equation (*with velocity* c) $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{4}$$

The first two equations involve $\partial/\partial t$ (first order) so initial values $u(x,0)$ will be given (at $t = 0$). We know where the solution starts, and in Figure 3.1 those initial values are delta functions. Notice the difference at $t = 1$! In the one way wave equation, the delta function moved to the left. In the heat equation, the delta function diffused into a Gaussian. And it spreads out even further by the time $t = 2$.

Since the initial value is symmetric around the centerpoint $x = 0$, so is the solution $u(x, t)$. The heat equation doesn't notice if you change x to $-x$, but it sure notices if you switch t to $-t$. The "backward heat equation" $-\partial u/\partial t = \partial^2 u/\partial x^2$ is impossible to solve. Physically, hot air can spread into a room, but time doesn't reverse and the diffused heat doesn't return back to the starting point.

The full wave equation involves $\partial^2 u/\partial t^2$, so we need an initial velocity $\partial u/\partial t(x, 0)$ in addition to $u(x, 0)$. In Figure 3.2a that initial velocity is zero. We see waves going in both directions (symmetrically). In Figure 3.2b the initial velocity is $\partial u/\partial t(x, 0) = 1$. The wave to the left is different from the wave to the right. You might note that the same word "velocity" was used for the number c (velocity in $x - t$ space) and for $\partial u/\partial t$ (velocity in $u - t$ space).

How could those examples be extended? The one way wave equation could become

$$???\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\partial u}{\partial t} = c(x)\frac{\partial u}{\partial x} \quad \text{or even} \quad \frac{\partial u}{\partial t} = c(u)\frac{\partial u}{\partial x}\,. \tag{5}$$

The last of those is nonlinear! It is highly important, one good application is to traffic flow. At a point x on the highway, the car density is $u(x, t)$ at time t. If the density up ahead (one way drivers!) is greater, then cars slow down and get denser. The relation depends on u itself, it is not linear. This produces the waves of stop and go driving that a helicopter sees in a traffic jam.

The heat equation should have a "diffusivity constant" c, with the dimension of $(\text{distance})^2/\text{time}$. In fact this fits our framework exactly, there is a perfect analogy with $K = A^{\mathrm{T}}CA$ and $u' = -Ku$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(c(x)\frac{\partial u}{\partial x}\right). \tag{6}$$

When $c(x)$ is a positive constant, we can rescale time to make $c = 1$. That is the case we can solve. (When c is a negative constant, nobody can solve the backward heat equation. We never allowed $c < 0$ in Chapters 1 and 2 either.) When c depends on u or $\partial u/\partial x$, the equation becomes nonlinear and we don't expect an exact formula (but we can compute!).

??? The wave equation would also look better in its symmetric form using $A^{\mathrm{T}}CA$. Notice also that it can be rewritten as

$$\frac{\partial}{\partial t}\begin{bmatrix} \frac{\partial u}{\partial t} \\ c\frac{\partial u}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} \frac{\partial u}{\partial t} \\ c\frac{\partial u}{\partial x} \end{bmatrix}. \tag{7}$$

In a sense (Problem A) this is a pair of one way wave equations!

Those time-dependent wave and heat equations will come after we study the all-important equation of equilibrium: **Laplace's equation**. This describes a steady state. The variables are x, y, z (in three space dimensions) or x and y (in two dimensions—we will concentrate on this very remarkable model).

Laplace's equation has pure second derivatives $\partial^2 u/\partial x^2 = u_{xx}$ and $\partial^2 u/\partial y^2 = u_{yy}$:

$$\textbf{Laplace:} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{d^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0 \quad \text{in a plane region } R \tag{8}$$

This describes the steady state temperature $u(x, y)$ over the region R, when there is no heat source inside (the right side of the equation is zero). The problem doesn't have initial conditions, it has boundary conditions! The boundary of R is a closed curve C. At every point of C we may prescribe *either* a fixed temperature u_0 *or* a heat flux F_0:

$$\text{Boundary conditions:} \qquad u = u_0 \text{ or } \frac{\partial u}{\partial n} = F_0 \quad \text{at each point of } C\,. \tag{9}$$

That "normal derivative" $\frac{\partial u}{\partial n}$ is the rate of change of u in the direction perpendicular to the boundary. At a point where the boundary is insulated (meaning that no heat can flow through) the flux is $\partial u/\partial n = 0$.

This is the problem of Section 3.2: Laplace's equation (8) with boundary conditions (9). It is the two-dimensional analogue, a partial differential equation, of the most basic two-point value problem:

$$\frac{d^2 u}{dx^2} = 0 \quad \text{with} \quad [u(0) \text{ or } u'(0)] \text{ and } [u(1) \text{ or } u'(1)] \quad \text{given at the endpoints}\,. \tag{10}$$

This describes the displacement (or it could be the temperature) in a rod. The solution to equation (10) is just $u(x) = A + Bx$. For Laplace's equation we will list an infinite family of solutions (which we need because there are infinitely many more boundary points!).

Equation (10) was our simple model, with no applied force f and with a constant coefficient $c = 1$. The more general form in Chapter 2 was

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x), \text{ with boundary conditions on } u \text{ or } w = c\frac{du}{dx}\,. \tag{11}$$

Those possibilities for f and c are also seen in two dimensions. When there is a source term $f(x, y)$ we have **Poisson's equation** (pronounced Pwa-son):

$$\textbf{Poisson:} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = f(x, y) \quad \text{in the region } R\,. \tag{12}$$

When the material in R is not homogeneous, the constant coefficient $c = 1$ becomes a variable coefficient $c(x, y)$:

$$\textbf{Nonhomogeneous:} \ \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) = f(x, y) \quad \text{in the region } R \tag{13}$$

Maybe you can see that we are closing in on our favorite framework $A^{\mathrm{T}}CAu = f$!

Section 3.2 sets this framework, by identifying A and A^{T}. Those are the key operators of vector calculus, the gradient and the divergence. Laplace's equation, with $c = 1$ and $f = 0$, is seen as div grad $u = 0$. Then we concentrate on solving this exceptional equation, by analysis or by scientific computing:

Exact solution (formula and series): Section 3.2
Numerical solution (finite differences and finite elements): Section 3.3.