CASEY RODRIGUEZ: We'll continue our discussion of the Riemann integral. So let me just recall a few bits of notation from last time and then also, the main result that we proved at the end of last class. So we had partitions, which are just finite sets. So here, a is less than b, and we're looking at an interval ab. Partitions, they're just finite sets of ab. And then we also had tags, which were finite sets as well with c one between x0 and x1, which is less than or equal to xi 2, which is less than or equal to x2 and so on.

And we call these tag partitions. And we also had the norm of a partition. This is the max of-- so we think of partitions as being-- as breaking up the interval ab up into smaller subintervals, x0 up to x1. I got to get a different piece of chalk. This one is going psychedelic on me.

And then the xi ones are just points chosen in these smaller subintervals. The norm of a partition is the length of the largest subinterval. And then we had the Riemann sum associated to a partition, tag partition xi, which is the sum from j equals 1 to n, f xi j xj minus xj minus 1. OK? This just some notation from last time.

And then what did we prove? We proved the following theorem, which is the existence for the Riemann integral, which is the following that for all continuous functions on ab, they exist a unique number, which we denote integral abf with the property that-- with the following property. That if I take any sequence of tag partitions with norm going to 0-- so this is the partitions are getting finer and finer. They're all a sequence of tag partitions with norms going to zero, we have the associated Riemann sums, this sequence converges. So this is now a sequence of real numbers, and equals this number, integral abf.

No matter what sequence of tag partitions I take with norms converging to 0, the Riemann sums converge to this number, which we call the Riemann integral of f. Net, I think the very last thing we proved was that the Riemann integral is linear. The integral of the sum is the sum of the integrals. And if I multiply a continuous function by a scalar, then the scalar pulls out. OK?

So now we're going to discuss some properties of the Riemann integral. So the next is that in some sense, the area of the union, if you like, is the sum of the areas if we think of the integral as being the area underneath the curve. So this next property is the additivity of the Riemann integral, which states the following-- if f is in cab, and I take a point between c and b, then the integral from a to b of f is equal to the integral from a to c of f plus the integral from c to b of f.

So to prove it, again, all we know about these numbers is that they satisfy this property. So what we'll do is, we'll take a sequence of tag partitions of ac converging to this integral in a sequence of tag partitions from c to b converging to this integral when I stick them into the Riemann sums. But then when I take the union of those tag partitions, I get a partition of ab, which will approach the integral of ab. That's the basic intuition.

And of course, what was I saying a minute ago about area of the union being sum of the areas? Well again, like I said, if we think of the Riemann integral as being the area underneath the curve, then this says that the area from a to b underneath the curve is equal to the area underneath the curve from a to c, plus the area underneath the curve from c to b.

OK, so let's take-- someone needs to write notation here. Let's take two partitions. Let be a sequence of partitions-- so a sequence of tag partitions, I should say-- of ac such that norm of this partition goes to 0. So this is now a partition of ac. And then we'll take another sequence of tag partitions of cb.

For those watching at home wondering what these letters are, that's the Greek eta. That's the Greek zeta. So we have a sequence of partitions of ac. These are the y and eta. And then we also have a sequence of partitions of cb such that the norm of these guys go to 0 as well as r goes to infinity.

So now if I have a partition of ac and a partition of cb, if I take their union, then I will get a partition of ab. This will be the union of these guys. So it's easy to see that this is now a sequence of tag partitions of ab. And moreover, so what is the norm of this new tag partition? What is the maximum length of the subintervals? Well, it's just going to be the maximum of the norms of y and z.

And note that the norm of x sub r, this is equal to the maximum of the norm of yr and the norm of zr. And since both of these converges to zero, the maximum of them also converges to zero.

So now we just put this all together. So by this theorem, what do we know? We know that the Riemann sum associated to this converges to the integral ab of f. The Riemann sum associated to this sequence of tag partitions converges to acf. And then, this sequence of Riemann sums associated to these tag partitions converge to cbf. OK? And now we just use the fact that taking a Riemann sum is additive, right?

Since the Riemann sum associated to this partition, which is just a union of these partitions, is equal to-- this implies taking the limit as r goes to infinity. The left hand side, as I wrote down here, converges to abf. And then the right hand side, the limit of the sum is the sum of the limits. So this converges to acf. And then the right thing here converges to-- the second thing here converges to the integral from c to b of f. And that's it.

So really, the fact that the integral is additive follows from the simple fact that Riemann sums are additive, and the fact that limits respect algebraic operations. The limit of the sum is the sum of the limits.

All right, so the Riemann integral is additive and also linear. So we see that the Riemann integral is something that respects algebraic operations with respect to the real numbers namely, adding and scalar multiplication. And it respects I guess, what you could say topological considerations. Or, not really topological, but this additive property is a very natural property. What am I getting at here? So now one could ask, how does the integral interact with inequality?

So this is something we ask about all limits, right? Or at least limits of sequences that we saw. If one sequence lies below another one, taking limits respects that. Of course, that's not true for the derivative. I can have two functions, one bigger than the other, and the derivative of the smaller one be bigger than the derivative of the first one. But so one can ask the question let's suppose I have two functions, one smaller than the other. What is the-- what's the relationship between the integrals?

OK. So suppose f and g are now two continuous functions on ab. So then the first part of this theorem is that if for all x and ab f of x is less than or equal to g of x, then the integral from ab of f is less than or equal to the integral from ab of g. All right? And this is kind of understandable. I mean again, if we think of the integral as being a theory of the area underneath the curve. If I have one function sitting above another function, then the area should be bigger than the area of the smaller one underneath the curve, which sits below the first one.

And now the Riemann integral is a limit of sums in some sense, and a limit of certain sums involving f. And what? So we do have this relationship involving sums and absolute values with the triangle inequality, which says the absolute value of the sum is less than or equal to the sum of the absolute values. And if you think of the integral as being essentially in a limit of sums, or if you like, a continuous sum, then you should expect that the absolute value of the integral is less than or equal to the integral of the absolute value. And that's indeed what we have.

Second is the absolute value of the integral from ab of f is less than or equal to the integral from ab of the absolute value of f. Like this is the triangle inequality for integrals.

All right, so we'll prove one, using again, the main property we have about Riemann integrals that they satisfy this property. And then we'll deduce two directly from number one. So let's take a sequence of tag partitions with norms converging to zero. Then for all r, if we take the Riemann sum with respect to f of this tag partition, this is equal to j equals 1 to n. This is just the definition.

r here is just indexing the sequence of tag partitions. And I should say this doesn't have to end at some single number-- in fact, it can't-- in common with all of these tag partitions. So here n of r is just the number of partition points that we have in our partition x superscript r. And since we're assuming f of x is less than or equal to g of x for all x and ab, this is less than or equal to sum from j equals 1 to n of r, g of cj of r.

OK, and what is this last thing? This last thing is just equal to the Riemann sum of g, with respect to this tag partition x and xi. So we started off with the Riemann sum associated to f, and we showed that's less than or equal to-- just kind of summarizing here. That this Riemann sum is less than or equal to this Riemann sum.

So now if I take the limit as r goes to infinity, this on the left hand side approaches the integral from ab. And since limits respect inequalities, I get this, which is what we wanted to prove.

All right, and from number two-- I mean, for number two, we get from number one-- I mean, we get number two from number one, essentially quite quickly. So since plus or minus f is less than or equal to the absolute value of f, we get that plus or minus-- so integral of plus or minus f is less than or equal to the integral from ab of the absolute value. Now scalar multiplication we proved last time pulls out.

So that means plus or minus the integral of f is less than or equal to the integral of the absolute value of f, which is the same as-- so for the minus sign this tells me this inequality. And for the plus sign, I get that inequality. But that's equivalent to saying that the absolute value of the integral of f is less than or equal to the integral of the absolute value.

And that's the proof of the theorem. Let me make a small remark here-- and this will be an exercise on the homework to prove this-- is the following. We know that taking limits respects inequalities but not necessarily strict inequalities. So I can have two sequences, one strictly less than the other for all n. And then as n goes to infinity, their limits could in fact equal each other. So for example, if I take the first sequence to be 0, the second sequence to be 1 over n, 1 over n is always bigger than zero. Yet, they both converge to 0.

So taking limits does not respect strict inequality. However, in this setting, which is very nice-- I mean, it's actually an extremely important property of the integral is that this limiting process that we do to take integrals in fact does-- it's not just any old limiting process. It actually does respect strict inequality. So what am I saying? Is that we can prove something a little bit stronger. And that's-- which I'll say it in its kind of reduced form here.

If I take a continuous function and it's positive, then we already know that the integral of f is going to be non-negative by what we just proved. But in fact, the integral of f is positive. And again, this is also kind of important from a psychological standpoint. If we're thinking of the integral as being a theory of the area underneath the curve. So what this states is that-- and again, I'm not going to prove this. This will be on the assignment.

Again, this coincides with our intuition of what a theory of the area underneath the curve should be. So this says if a continuous function is positive-- like the picture I drew you-- then the integral, which we interpret again, as the area underneath the curve, is positive. So that matches what our desire for this to be a theory of the area underneath the curve should be. All right?

And so we have- if you like-- I mean, we could approve this from basic considerations in what the definition of the integral is. But we have the following theorem that for all alpha and r-- or I could just say the following. Well, maybe not stated this way.

Going a little off script here, but it's OK. So we actually haven't computed a single integral yet. Really, it's tough to with this definition. In a minute, we'll prove the fundamental theorem of calculus and be able to do it very easily. But let me first prove at least the most basic integral. So integral from ab of the function 1 equals b minus a.

So what's the proof? Take a sequence of partitions with the norms converging to 0. And we just look at the Riemann sum. And for one this will be quite easy to compute what the Riemann sums are. So this is a sequence of partitions with norm converging to 0. Then if I look at associated Riemann sums, this is equal to j equals 1 to n of r.

No matter what I-- no matter what xi is, I just get 1, and then times the length of the interval, the subinterval. Now this is a telescoping sum. Meaning this is equal to x1 minus x0, plus x2, minus x1, plus up until xn minus xn minus 1. So all I pick up is-- not x1-- but x0 and xn. So this is equal to xj of r xn, sorry, of r minus x0 of r. And this is the last point in the partition. This is the first point in the partition. And as always for partitions, the last point is b. The first point is a. So this is equal to b minus a.

Now this thing on the left hand side is equal to b minus a. So this tells me that the integral, which is equal to this limit of Riemann sums is equal to b minus a. All right, and then the assignment for this-- or I guess, when you see this last week, was to essentially, compute the integral for-- the integral from ab of x, dx.

And so we get from this theorem and this theorem following bound for the area. So let me draw a picture so that this is not so surprising. Let's say I have a function from a b. There's the function f. So that was supposed to go through this point. So let's make that a little bit-- I suppose I don't need these parts.

Let's say I have-- so far a continuous function, it always achieves a minimum and maximum. And at least for this picture, the minimum occurs here. The maximum occurs here. Now, what's the comparison about areas from this picture? Well, the area underneath the curve of f will be bigger than or equal to the area underneath the curve of the constant function equal to little m sub f. And that's just b times a times m sub f, again, by intuition. But this is also seen from the theorem we just proved along with the linearity of the Riemann integral, which we've already proved.

And the integral of f, the area underneath this curve is also less than or equal to the area underneath the curve where f achieves a maximum. It's the 12th week of class, and I still haven't brought colored chalk, so hopefully, that was clear enough without me having to color stuff in.

So the theorem is the following. If f-- again, we're working with continuous functions because that's all we can integrate or are going to integrate in this section-- and I have these two numbers, the inf of fx, which we know is in fact, a min, meaning there exist a point in here-- so in fact, I'm going to write "min." we know that by the min, max theorem that for every continuous function it achieves its minimum-- and max.

Then, as I was saying a minute ago, the area underneath the curve little m sub f, which is this smaller square, is less than or equal to the area underneath the curve of f. So little m sub f times b minus a, the area underneath the minimum, is less than or equal to the area underneath the curve of f, which is less than or equal to the area underneath the curve of the taller line, capital M sub f.

How do we prove this? Well, we just use these previous two theorems, right? Since little m sub f is less than or equal to capital M sub f-- so since little m sub f is less than or equal to f of x, which is less than or equal to capital M sub f, for all x and ab we can apply that theorem up there to get that the integral of little m sub f-- again, this is just a fixed number-- is less than or equal to the integral from ab of f.

And now again, these are just fixed numbers. So by what we proved here that the integral from a to b of 1 is b minus a, the fact that scalers pull out-- so this number here can come out-- and this is just m sub f times the integral from ab of one. So this is little m sub f times b minus a. Over here, this is capital M sub f times b minus a. And that's the proof.

So now let me make a few comments about some conventions, really. So we've been talking about-- or at least, I've been talking about, you're not here. So we haven't been talking at all. So we've been talking about the integral from ab to f when a is less than b, But I'm going to use-- I'm just going to set down some convention, or you could say this is really notation.

When I have this-- when I write down this symbol, integral a to a of f, this is really just another way of writing 0. And another is that if, in fact, b is less than a, then the integral from ab to f-- so remember, we've been talking about the integrals from-- over intervals-- or where the number on the bottom is less than the number on the top, so whenever I write this symbol with the bigger number on the bottom and the smaller number on the top, you should read this as minus the integral of, with this being in the right place, meaning this number here is less than this number here.

OK and so again, these are really just notations, that if I write this, this is a fancy way of writing 0. When I write this, this means minus the integral with the right numbers on top and bottom, meaning the smaller numbers on the bottom, the bigger numbers on the top.

So maybe you ask-- so this is-- so why do I define that to be 0. Well, if you like, 1 is consistent with the fact that for all continuous functions, if I take the integral from a to b with b bigger than a, and I take the limit as b goes to a, this equals 0. And two is then consistent with number one and then additivity.

With additivity in number one, because then by number one, I get that the-- so additivity will tell me that the integral from a to a of f is equal to the integral from ab to f, plus the integral from baf, here assuming b is less than a. So if by 1 this is 0, then to be consistent with additivity, this would force the integral from ab to f to be minus the integral from ba of f. So both of these conventions, if you like, are consistent with the properties of the Riemann integral, namely that the limit as b goes to a of the area underneath this curve, as the base gets smaller and smaller is 0. And then from-- if I assume this, then additivity tells me that the integral from ab has to be minus the integral from b to a of f.

All right, so now we computed successfully one integral. Of course, when you took calculus, that was not all you could do. Calculus is what it is because of the hero of the story, which is Fundamental Theorem of Calculus, which is what we'll prove now. Fundamental Theorem of Calculus, which states following. If I have a function big F-- so first off, let little f be continuous function.

So the first statement is basically how to compute integrals. If capital F integral from ab to r is differentiable everywhere on ab and f prime equals little f, then the integral from ab of little f equals f of b minus f of a. Another way of saying this is that the integral from ab of capital F prime is equal to f of b minus f of a.

And then the second part-- so the first part is about computing integrals. The second part is about solving differential equations, basically. So the function g of x equals the integral from a to x of f. So for each x and ab, I stick it in as the upper limit on this integral. This function is differentiable on ab. And it satisfies the simplest differential equation, which is g prime equals f, and g of a equals 0.

So this is the Fundamental Theorem of Calculus, the first part being about how do you compute integrals, the second, how do you compute solutions to a differential equation. Namely, how do I find the solution to the problem g prime equals little f. So little f is the given. I want to find a function g that satisfies g prime equals little f with initial condition capital G of a equals 0. That's given by this function here where I take little left and I integrate it from a to x. So that's how I interpret the Fundamental Theorem of Calculus, the first about solving integrals, the second about solving differential equations.

All right so this is the hero of calculus, the Batman of the story. And I said at one point that the Mean Value Theorem is the Alfred of this story. And therefore, it should play a decisive role in proving this theorem. All right. So we're going to connect these two things. So in order to even get at this guy, we have to take a partition, a sequence of partitions converging with norm converging to 0. So and then see what we get.

So this is the proof part one. So let's take a sequence of tag partitions-- I keep forgetting to write tag and say the word tag, but you should hear that even though I don't say it-- with norm. Actually, we're not even going to-- this is correct. I'm actually going to come up with the tags in a minute. So let's first take a sequence of partition. So this is actually completely correct. So no tags yet, no points chosen in between, with Norm converging to 0. And then I'm going to choose a special tag for each r so that in the limit I get this equality between the integral of f and capital F of b minus f, capital F of a.

So here comes Alfred by the Mean Value Theorem. So we have this subinterval x of j minus 1 and x sub j. So by the Mean Value Theorem, there exist-- for each j there exists a point in between these two guys such that if I take f of x sub j r minus f-- so capital F, remember, is this function that's differential and whose derivative gives me little f-- this is equal to capital F prime of xi j of r times xj minus xj minus 1.

And by assumption, the derivative of capital F is equal to little f, right? So I'm going to replace this f prime with little f. So now these xi's I will take as my tag, my sequence of tags. Put a star by this relation here. What do we conclude? If I look at the Riemann sum for f associated to now this sequence of tags, where now the xi's are exactly these xi's that satisfy this relation.

This is equal to as before, sum from j equals 1 to nr f of xg xi sub j. Now by how these xi's were chosen-- remember, they were chosen to satisfy this relation here, star by the Mean Value Theorem. So this is equal to the sum from j equals 1 two n sub r, f of x sub j minus f of x sub j minus 1. And again, this is a telescoping sum. And so all I pick up is when j equals n of r, the last point. And when j equals 0-- so this is equal to f of the last point minus f of the first point. Now for partition, this is always b. And this is always a. So this is f of b minus f of a.

So every one of these Riemann sums for this sequence of tag partitions gives me f of b minus f of a So then I get to take the limit, and I get the integral. And therefore, integral from ab to f, which is limited as r goes to infinity of s of f equals-- and we just computed this is always equal to f of b minus f of a.

We took a sequence of-- just a recap-- we took a sequence of partitions with norms converging to 0. And we chose special tags using the Mean Value Theorem. So by the Mean Value Theorem, on each one of these subintervals capital F evaluated at the right endpoint minus f evaluated at the left endpoint is equal to f, which is the derivative of capital F evaluated at some point in between times the length of the interval.

So now if we take the Riemann sum associated to this sequence of tag partitions where these xi's are defined by this condition, then we can actually compute this Riemann sum, and it just gives me f of b minus f of a no matter what r is. So then when I take the limit as r goes to infinity, the left side goes to the integral, but it's always equal to capital F of b minus f of a. So that proves number one.

So for number two, what do we want to show? We want to show this function. So first off, it's by convention g of a, which is the integral from a to a of f, this is zero. So I don't have to check the second condition. I just need to check that this function is differentiable, and the derivative of that function is little f.

Let c be in ab. So what do we want to show? We want to show the derivative of capital G is equal to little f. So we would like to show that the limit as x goes to c of the integral from a to x of f minus integral from a to c of f, over x minus c, which is just g of x minus g of c, equals f of c. so we're going to do this by the books-- by the book, I guess. When I say-- so that's kind of a bad pun, I guess. I don't know. But meaning, an epsilon delta proof for this limit. So let epsilon be positive. The Delta that we pick in the end depends on-- and we'll use crucially the fact that f is continuous at c.

So since f is continuous at c, there exists a delta 0 positive such that what? t minus c less than delta 0 implies that f of t minus f of c is less than epsilon-- let's give ourselves a little room-- over 2. Choose delta to be this delta 0. Remember, we have to show for every epsilon, there exists a delta so that if x minus c is less than delta, then this minus this in absolute value is less than epsilon. I'm now saying, choose delta to be this delta 0 where this delta 0 is ensuring this right here. OK as long as t minus C is less than delta 0.

Now because of sign conventions, we'll do two cases. So not really, I'm just going to do one case. So suppose that x is between c and c plus delta. So we want to also show-- we'll also need to be able to do the case that x is between c minus delta and c. That would cover the whole range of the absolute value of x minus c is less than delta but bigger than zero. I'm just going to do this case.

So now we want to show that if x is in here, then this thing minus this thing in absolute value is less than epsilon. So first off, I would like to note that if t is in this interval c to x, then that tells me that the absolute value of t minus c, which is equal just t minus c, this is less than or equal to x minus c, which is less than delta. And remember, this equals delta 0, all right?

So as long as I'm in this interval here, the absolute value of t minus c is less than delta 0. So let me just draw a picture so that this is pretty clear. This length is delta. If I take any t in between, then this distance from t to c is also going to be less than delta. So that's all I've written down.

Thus, by now compute the difference quotient minus the proposed limit. So looking at 1 over x minus c times the integral minus the integral from ac of f minus f of c. This is the thing I want to show is less than epsilon. Now by additivity, the integral from a to x since x is-- well, the integral from a to x of f is equal to the integral from a to c plus the integral from c to x. So this is equal to 1 over x minus c integral from c to x of f minus fc.

Now, this is just a fixed number. And therefore, I'm going to do a little trick. Minus the integral from-- so in fact, let me do this. Let me throw in some integration variables so that this becomes clearer. So this is f of t, dt minus integral of f of t, dt. I've been using the notation where I just don't write down the integration variable, but let me do that here. F of t, dt minus f of c.

Now this is equal to minus f of c over x minus c integral from c to x of 1 dt. Because the integral from c to x of the constant 1 gives me x minus c. That cancels with this guy. Now f of c is just a number. It's just a fixed number. So I can bring this inside the integral. And then use the linearity of the integral to in fact, rewrite all of this as 1 over the absolute value of x minus c integral from c to x of f of t minus f of c, dt in absolute value.

Now 1 over x minus c, this is positive because x is bigger than c. So I can pull this out just like that. And now what do I know? For t in between this c and x, the absolute value of t minus c is less than delta 0, right? So by what I have here, as long as I'm in this interval cx, I'm going to have this inequality hold. So before I do that, let me first apply the triangle inequality for integrals. This is less than or equal to the integral from c to x of f of t minus f of c.

So by the second thing that I've underlined, for t in this interval c to x, this thing here is always less than epsilon over 2. So by what we know about integrals respecting inequalities, this is going to be less than or equal to 1 over integral x minus c, integral c to x, epsilon over 2 dt. And now this is just equal to-- so this is a number. It comes out. So I get 1 integral from c to x of 1. So I get x minus c times the integral of 1. So x minus c, and that cancels, less than epsilon.

And this was for x between c and c plus delta. The argument for c between c and c minus delta is essentially the same, just with minor sign changes. So similarly, c minus delta less than c implies that the integral from a to x of f minus integral a to c of f over x minus c minus f of c is less than epsilon. So we've done x between c minus delta and c, x between c and c plus delta, which implies that if-- then, which is what we wanted to prove.

OK, so we've concluded the proof of the Batman of the story. And this gives us not only a way of computing integrals, but useful way of, if you like, shifting the burden or shifting the blame whenever we do compute the integrals of certain products. So what I'm talking about is the integration by parts, which I can't remember if I said it was first or second. I think he was second. First most useful thing from analysis is the triangle inequality, the second being integration by parts. And a close third being the Cauchy-Schwartz inequality, which in conjunction with integration by parts is quite a-- most current research papers are based on these two things and using them in a very clever way.

So integration by parts is suppose f and g are continuously differentiable, what that means is that the derivatives exists on ab, and they're also continuous. So if you hear me say continuously differentiable in the future that means that the function has a derivative that's continuous. Then the integral from ab of f prime times g-- so the one with the burden, because remember, differentiablity is kind of a miracle. So miracles always come with some sort of burden. This is equal to f of b times g of b minus f of a times g of a minus the integral from a to b. And now we shift that burden to g times g prime.

So let me put a parentheses around that so you can-- I don't-- let's do it this way, g prime f. All right, so I can take that derivative and put it on g. And what's the proof? The proof is just the Fundamental Theorem of Calculus and the product rule. Since the derivative of f times g is equal to f prime times g plus g prime times f, we get by the Fundamental Theorem of Calculus the integral from ab of, let's say the right hand side, so this is f prime g plus g prime f is equal to the integral from a to b of f times g prime, so by Fundamental Theorem of Calculus.

And now the integral of the derivative is the function evaluated at the endpoint. So that's f of b times g of b minus f of a times g of a. So that's integration by parts, which is a consequence of the product rule. the quotient rule is something-- is just really the product rule in disguise kind of. So we don't get really anything new from there.

But we also remember, have the chain rule, which then results in the change of variables formula. So theorem-- let b from ab to cd be continuously differentiable. So this means the function is continuous, and its derivative is also continuous. With the property that the derivative is positive on ab. b of a equals c. b of b equals d. It's hard to see that a.

So you can think of phi as being a change of variables from cd to ab. Then the integral from c to d of f of u, du is equal to the integral from a to b of f of phi of x, phi prime of x, dx. So maybe instead of change of variables-- so this is change of variables, a.k.a. u substitution, where you said u equal to phi of x, then du is equal to phi prime of x dx.

And again, the proof of this just follows from the Fundamental Theorem of Calculus and the chain rule that we know. So let f be a function from ab to r such that f prime equals little f. We can always find one by the second part of the Fundamental Theorem of Calculus. We can always solve this differential equation up to a constant. Then if I look at the function f of phi of x, and I take its derivative with respect to x by the chain rule, this is equal to f prime of phi of x times phi prime of x, which equals f of phi of x, phi prime of x.

So then when I integrate that, integral from ab, phi of x-- f of phi of x phi prime of x dx, this is equal to the integral from ab of f of phi of x prime dx. Now by part one of the Fundamental Theorem of Calculus, this is equal to-- the integral of the derivative is this thing evaluated at the endpoint. So f of phi of b minus f of phi of a. This is equal to f of d minus f of c because phi of b equals d, phi of a equals c, and phi of b equals d.

But again, by the Fundamental Theorem of Calculus since capital F is-- when you differentiate, it gives me little f, this is equal to the integral from c to d of f prime of u du, which is equal to f of u du. Again, so we just applied the Fundamental Theorem of Calculus really, three times, OK? First to find a function whose derivative is little f. Then we used the chain rule to conclude that the derivative of capital F of phi equals this thing, which is the thing we're integrating on the right hand side.

So then when we integrate that, that's equal to the integral of a derivative. And therefore, we just pick up the endpoints, f of d minus f of c. But because capital F is equal to-- is an anti-derivative of little f, meaning it's derivative is little f, this number here is also equal to this integral here, which is this guy. And so then we get the change of variables formula.

So I think we'll stop there. And next time, we'll do a quick application of the second most useful thing on Earth, which is integration by parts. And then we'll move on to sequences of functions.