[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIGUEZ: OK. So we're going to continue our discussion of-- is it 1M or 2? It's 1. Of the Riemann integral, which, remember, from the discussion at the end of the last lecture, is a theory of area underneath the graph of a function.

So what is that theory? The theory is built up as follows. Given a continuous function f-- so this is just a refresher on the definitions I introduced last time-- a partition is just points between-- just breaking up the interval from a to b into points, into little subintervals, with the norm of this partition being the length of the longest subinterval [INAUDIBLE] which we refer to as a tag.

It's just a set of points with one lying in each of these subintervals-- so for me, let's take them all to be the right endpoint of these intervals-- and then associated to a tagged partition, which is a pair, x and [? xe. ?] We associated a Riemann sum, which is f evaluated at these points times the length of the interval.

So if we draw the graph of a function f and-- say, in this picture-- this number this Riemann sum would then be equal to the area of these three boxes here. OK? And so our theory of Riemann integration, or our theory of the area underneath the curve, is built on the following goal, which is to show that, as these partitions get finer and finer, as the norm of these partitions goes to 0, these Riemann sums should converge to some number.

And that number, which we call the Riemann integral of f, we interpret as the area underneath the graph of f. So the first goal-- the main goal, really, to start off with-- is to show that this is a reasonable-- that this is actually true that, these Riemann sums do, in fact, converge to some number as they become finer and finer for a given continuous function.

So that is the main theorem, which I think is going to be the main thing that we prove today. So theorem of the Riemann integral, which is the following-- let f be a continuous function from a b to r-- which, remember, this is the notation we used from last time. Let me move my picture over here.

Then there exists a unique real number, which we denote as the usual symbol, integral a b f of x dx with the following property-- for every sequence of partitions xr cr, such that the norm of these partitions is going to 0 as r goes to infinity-- so these partitions are getting finer and finer if the norm is going to 0.

Remember, the norm is the length of the longest subinterval I have. So for all sequence of partitions, with norms converging to 0, we have that the limit of the Riemann sums exists and equals this number, which we refer to as the Riemann integral of f. OK? So there's a lot to unpack here.

First off, what this number, integral from a b to f of x dx-- this property that it has is, no matter what sequence of partitions you take, as long as they're becoming finer and finer, this limit actually exist, and it equals the same number. It equals this number, integral a b f of x dx. So I could take two different sequences of partitions becoming finer and finer, look at the Riemann sums, and those two sequences of Riemann sums converge, and they converge to the same number-- again, which we denote by integral a b f of x dx, the Riemann integral of f. And so it's this number, which we interpret as the area underneath the graph of f. OK?

All right, so our main goal for today is to prove this theorem. So first off, there's two parts of the statement here. There exists a unique real number, which we denoted by that with this property. Uniqueness is immediate. If I have two real numbers, such that, for all sequence of partitions with norm going to 0, the limit of the Riemann sums equals that limit, well, limits of sequences are unique, and therefore, those two numbers have to be the same.

So uniqueness is clear. It's really the existence part that we have to prove, that there exists a real number so that, no matter what sequence of partitions we take, the corresponding Riemann sums converges to 0-- as long as they become-- the partitions become finer and finer. OK? OK.

Now, we're not going to prove this theorem just yet. We need some facts that will be used in the proof, so we're going to put off proving this theorem for a few minutes and first prove some necessary facts. So let me first define a useful metric or number associated to-- or function associated to a continuous function called the modulus of continuity.

So for f continuous function and eta positive, we define the modulus continuity omega f of eta-- this is equal to the supremum of f of x minus f of y [INAUDIBLE] absolute value, such that the absolute value of x minus y is less than or equal to [? eta. ?] OK? So what does this modulus of continuity measure? It measures, given-- if I take any two points less than or equal to eta in distance, and I look f of x minus f of y in absolute value, and take the sup over all that, that gives me the modulus of continuity.

Now, for example, let's do the simplest example possible, f of x equals ax plus b. Then what do we get? f of x minus f of y-- this is equal to a times x minus y. So if x minus y is less than or equal to eta, we get that f of x minus f of y in absolute value is less than or equal to-- well, it's equal to a times x minus y is less than or equal to a times eta. Right?

And therefore, absolute value of a times eta is an upper bound for this set for this function. And it's also achieved. If I take absolute value of x minus y equal to eta, then this will be equality. So what I'm saying is that, for this example, for f equals ax plus b, the modulus of continuity is equal to a absolute value of a times eta. OK.

Now, something to note here is that, for this guy right here, as eta goes to 0, the modulus of continuity goes to 0 as well. And this is not special to this function. In fact, it's true for all continuous functions. So let me state the theorem and then give you an interpretation of it. So let me call this theorem 1.

So for all f continuous on the closed and bounded interval in a, b, limit as eta goes to 0 of omega f of eta equals 0. OK. So let me make one more comment here. Maybe this will also-- just to connect to something we did earlier, which follows immediately from the definition of this guy, that for all x and y, this is true. OK?

And therefore, you should think of, I'm trying to make this small, then the modulus of continuity is something that controls how close f of x and f of y are together. So somehow continuity is controlled by this function, the modulus of continuity. So if this goes to 0, then-- as this goes to 0, this goes to 0, then somehow I'm saying f of x and f of y are very close together. So again, the modulus of continuity is a way of measuring continuity of a function. That's one way to think of it.

All right, so let's prove this. This is a limited statement, and what we're going to do is we'll just prove it the old-fashioned way. Let epsilon be positive. What do we know about functions which are continuous on a closed and bounded interval? While, they're also uniformly continuous. So since f is continuous on this closed and bounded interval, f is uniformly continuous.

This means that there exists delta 0 positive, such that, for all x and y, x and y less than delta 0 implies that f of x minus f of y is less than-- let me put epsilon over 2. I'm going to give myself a little space here. OK? So just to recall, what we're trying to do here in terms of the definition of what this means-- we're trying to show that, for all epsilon positive, there exists delta positive, such that, for all eta-- eta's actually a positive number here, so-- implies omega of eta is less than epsilon. OK?

OK. All right, so this is what we're trying to prove. We know that, for a continuous function, it's uniformly continuous, and therefore, there exists is delta 0 so that, when x minus y is less than delta 0, f of x minus f of y is less than epsilon over 2 for all x, y in the interval. OK?

So basically, I'm going to choose delta to be this delta 0, the delta I need for this. And now let me show that it works. Suppose eta is less than delta, which, remember, is delta 0. Then, if x minus y is less than eta-- which, again, is less than-- less than or equal to eta, which is less than delta 0-- I get that f of x minus f of y is less than epsilon over 2.

Basically, I just rewrote what I had right there. And therefore, this implies that epsilon over 2 is an upper bound for the set f of x minus f of y, x minus y, which is less than eta, which implies that the supremum of this set-- which is, by definition-- so I'm just going to put brackets here, and I hope that's clear that I'm referring to this set here when I put that here-- that the supremum of this set has to be less than or equal to this upper bound, which is less than epsilon.

OK? So we've proven that, if eta is less than delta, then the modulus of continuity of f-- of eta is less than epsilon. And therefore, this is what we wanted to prove. OK.

All right, so that's one fact that I need, that this modulus of continuity-- which, remember, is the supremum of the difference of f of x and f of y, as long as x and y are bounded by eta-- that this converges to 0 as eta converges to 0 for all continuous functions. OK? And this is key in showing that the Riemann integral exists for continuous functions.

So theorem two is the following. So this is going to tell us how two Riemann sums are comparable if one partition is finer than the other. So if I have two tagged partitions of a, b, such that x prime is a subset of x-- so this means that x prime-- so this is a partition-- a breakup of a, b-- that contains all the points of x and more. OK? So it's a finer partition.

And we refer to x prime as a refinement of the partition x. All right, so we have two partitions. One is finer than the other, and a continuous function f. Then we can estimate the difference in the Riemann sums. This is less than or equal to the modulus of continuity of f evaluated at the norm of the courser partition times the length of the interval. OK?

OK, so this says, as long as-- one way to think about it is, if the partition x is very fine, and then I take a finer partition, that's not going to change the Riemann sum too much, because remember, this is converging to 0 as the norm of the partition-- how fine it is, or you can think how coarse it is-- is going to 0.

OK, so the idea is very simple. It's just going to take me a little bit just to write down. Like I said, this partition being contained in the other means this partition has all the points in x and then some more. We're going to break these partitions up into-- or at least this one up into parts where it's points in x and then plus the extra one.

So for k equals 1 to n-- so let me draw a picture first, and then I'll define the concepts that I have. So here we have the partition points of x. OK? And above I'm going to write now-- I used to write [? xe, ?] but I'm going to write the partition points corresponding to x prime. So this should be x prime sub l for some l, because remember, [INAUDIBLE] this one is contained in the other.

So this partition point will be x prime sub l for some l. And then I'll have some other ones, and so on, until I reach the last partition point from x prime, which is contained in this subinterval from the partition x. OK? So I let y with a k upper-- this is equal to the partition x sub k minus 1 equals x prime l plus 1, until I get to some partition point x sub m, which equals x sub k.

OK? So this is just a part of the partition x prime, contained in this subinterval, where these are partition points in x. And then I'm going to write eta k. This will be the tags that come with these points. OK. So what I want to draw attention to is that the tagged partition x, prime c prime-- this is equal to let me write it this way.

Since these are sets what I mean by-- let me write it as the union of the tagged partitions. And also, this eta has nothing to do with the eta from before. All right? I'm just using this as notation. So the full partition of x prime and [INAUDIBLE] prime tagged partition is equal to the union of these smaller partitions of xk minus 1 and xk as k runs from 1 to k. OK? I hope that's clear.

All right. All right, so now let's compare the part of the agreement Riemann for x [? xe ?] on this interval. So remember, somewhere sitting in here is the tag corresponding to the k-th tag for this guy. Let's compare it to the part of the Riemann sum for the x prime [? xe ?] prime also coming from this interval. OK.

In fact, let me make one more remark that, since this tagged partition is equal to the union of these tagged partitions, this implies that the Riemann sum corresponding to this tagged partition is equal to the sum of the Riemann sums corresponding to these partitions of these intervals, xk minus 1 and xk. All right? So these are not partitions of a, b, but they are partitions of xk minus 1 and xk. All right.

So I hope all this is clear. Again, I'm just breaking this Riemann sum up into parts where now I'm looking at what's happening between each of the partition points x sub k minus 1 and x sub k coming from the original courser partition x. All right?

Now we compute. So this is a part of the Riemann sum of x [? xe ?] from this interval minus the part of the agreement some for x prime [? xe ?] prime coming from this interval. Now, this is equal to f of [? xe ?] k. And so let me just rewrite this sum here. x sub k minus x sub k minus 1-- this is equal to sum from k equals, let's say, j equals l plus 1 to m x prime sub j minus [? x of ?] [INAUDIBLE] minus 1.

So of the smaller ones that are in there, this is just a telescoping sum. So all I pick up is x sub j prime sub m, which is x sub k, minus the first one, which is x sub-- x prime sub l, which is x sub k minus 1, minus-- and then this is, by definition, equal to j equals l plus 1 [INAUDIBLE] f of prime j, x prime j minus x minus 1-- and then a absolute value sign on the outside.

And then I combine these two, and-- this is equal to sum from j equals l plus 1 to k of k minus j times x prime j minus x prime j minus 1-- absolute value. Now, by the triangle inequality, the absolute value of the sum is less than or equal to the sum of the absolute value. So save myself writing-- I'm going to bring the absolute values inside.

OK. Now, [? xe ?] sub k and xe sub j-- they're both in the interval x sub k, x sub k minus 1. And by what I wrote up there, another way of-- or what follows simply from the definition of the modulus of continuity-- this is less than or equal to omega f of x sub k [? x sub ?] k minus 1 times x times j minus x prime j minus 1.

And this equals-- now, this doesn't depend on j, so this is, again just a telescoping sum. So I just pick up times x-- so that's for one piece, and now I just add them up. Then I look at-- and so like I said, this is equal to the sum of these terms that look like this from k equals 1 to n.

This is equal to the sum of terms like this from k equals 1 to n. That's what I wrote up there. And so if I apply the triangle inequality, this is less than or equal to sum from k equals 1 to n of f of [? xe ?] k times xk minus xk minus 1 minus s [? sub ?] f eta, which-- I have this estimate right here.

I've already estimated-- so started off with this. And I showed it was less than or equal to this. So now let me stick that estimate into here-- less than or equal to sum from k equals 1 to n omega f of xk minus xk minus 1 times xk minus xk minus 1. So another thing which you should notice about the modulus of continuity is, if I have something here and then something bigger, this set gets bigger, and therefore, this sup gets bigger.

So another property of the modulus of continuity is that it's increasing. This just follows from looking at the definition. There's more points to take a sup over f this thing gets bigger. So implies-- OK? So all of these lengths of these subintervals, the distance between partition points, those are all bounded by the norm of the partition, the biggest one.

So this is less than or equal to sum from k equals 1 to n omega f of norm of x times xk minus xk minus 1. And now, this doesn't depend on k, so I just pick up the sum from k equals 1 to n of this, which is a telescoping sum. All I pick up is x sub n minus x sub 0, which is just b minus a, which is the estimate we wanted to prove. OK?

OK, so now I'm going to do something that's sacrilegious, as far as board work goes. And I'm going to erase this board and prove the next theorem on it, because I want to keep what I prove-- or at least a statement of what I prove around for when we finally tackled the proof of the existence of the Riemann integral. So this is in permanent form on-- in digital form, and also in the notes.

Also, I should mention, we're not following the textbook now. So you have to read the lecture notes or watch the lecture. All right, so this theorem here, theorem two, allowed us to compare two Riemann sums, as long as one partition is finer than the other. But there's a simple trick which we can now use to prove how to compare any two Riemann sums for any two partitions.

So we have the following, theorem three. If xc and-- now they're any tagged partitions-- and f is a continuous function, then I look at the Riemann sum of this guy and then compare it to the Riemann sum of the other guy. This is less than or equal to the modulus of continuity evaluated at the first partition plus modulus of continuity applied to the second partition times the length of the interval.

OK? What does this say? This says that, if I take two very fine partitions, so that the norms are very small, and therefore, the modulus of continuity evaluated at these norms a small-- remember, the modulus of continuity converges to 0 as the argument goes to 0. So if I take two very fine partitions, then the Riemann sums are very close together. The Riemann sums barely change. And so now we have some hope to see that sequences of Riemann sum do, in fact, converge.

OK, so how do we prove this? It's a simple trick using the previous theorem to find a third partition to be the common refinement of these two. So take this partition and union it with the second one, and take a new set of tags to be the union of the two tags. So all the partition points of x and x prime-- I throw them together to get a new partition, x double prime-- and then the tags as well.

Then x-- backwards-- this new partition is finer than x. It's finer than x prime. And therefore, by theorem two, we have this estimate which we can use. So if I look at the difference of the Riemann sums, and I add and subtract the Riemann sum corresponding to x double prime and [? xe ?] double prime, and then I use the triangle inequality-- so I'm doing two steps here-- so add and subtract the Riemann sum corresponding to-- and then apply the triangle inequality.

For both of these, I can now insert this estimate here from theorem two, where-- what's the modulus of continuity being evaluated at? It's being evaluated at the courser partition. So x double prime is contained in x, so this is less than or equal to the modulus of continuity evaluated at x prime times b minus a. And then, for this one, x prime is coarser than x double prime. This one is contained in x prime-- so plus the modulus of continuity evaluated at x prime times b minus a. And that's what we wanted to prove.

OK. All right, so now we're in a position to prove this theorem of the Riemann integral. And so what we're going to do is we're first going to come up with a candidate for what the integral could be. And then we're going to show that's the actual-- that that candidate satisfies the conclusion-- or this property that, no matter what partition I take, with norms converging to 0, the Riemann sums converge to this number.

OK? So first we have to come up with a candidate number. All right. So first, take any-- so let's fix some partition of-- for the Greek letter aficionados out there, this is zeta. Let this be a partition of a, b with norm converting to 0 as r goes to infinity. You can always come up with some partition. So this [INAUDIBLE] tagged. You can always come up with one, at least one partition-- or sequence of partitions.

So this is [? be a ?] sequence of tagged partitions. OK, now this is finally right. So you can always come up with a sequence of tagged partitions of a, b with the norms converging to 0. Take the first one to be just the whole interval-- so just left endpoint, right endpoint. Next one add-- the midpoint. That's now three partition points. Now add the midpoint of the previous two intervals you had. Now add the midpoint of the intervals you had before, and so on, and that'll build up a sequence of partitions with norm converging to 0.

So I have this fixed sequence of tag partitions, and I claim that the Riemann sums converge for this guy. All right? So claim one-- the sequence of sums-- this is a sequence of numbers now-- OK. So it converges to some number. We're going to prove this by showing its Cauchy. Remember, Cauchy sequences of real numbers always converge. That's this completeness property of the real numbers.

OK. So there's no other way to do this. Then, using the definition-- the epsilon [? m ?] definition-- so let epsilon be positive. So by theorem one, there exists a delta positive, such that if eta is less than delta-- I should say-- so we know that the modulus of continuity for f converges to 0 as eta goes to 0. So for all eta less than delta, omega f of eta is less than epsilon over 2 times B minus a. OK? Let me put a star by this guy.

Now, since the norms of these partitions is-- are converging to 0, there exists natural number m0 such that, for all r bigger than or equal m0, the norm is less than delta. OK. And thus, for all r bigger than or equal to m0, if I look at the modulus of continuity evaluated at this norm of this partition-- so the norm of this partition is less than delta. And if I stick that into the modulus of continuity, anything I stick into the modulus of continuity which is less than delta-- which this is-- I should be less than epsilon over 2 times b minus a.

So really, this is the one I want to star, maybe not this one. I am using this one to get it, but this is the key point here. OK. So here we're just using the fact that modulus of continuity converges to 0 as the argument converges to 0, and that the norms are converging to 0. OK?

OK. So [? to show something's ?] Cauchy, we have to now choose m. I'm going to choose m to be m0. [INAUDIBLE] r, r prime bigger than or equal to m-- which, remember, we've chosen to m0. We have that the absolute value of s of f xr-- r, minus the Riemann sum, now with r prime-- absolute value. This is by theorem three, which we proved over there-- is bounded by the sum of the modulus of continuities evaluated at the norms times b minus a.

And now r and r prime are bigger than or equal to m0, and so by star-- so this line here. This is by theorem three. But now, by star, by our choice of m0, this is less than epsilon over 2 times b minus a plus epsilon over 2 b minus a times b minus a. This is now by star, which equals epsilon. And therefore, this sequence of Riemann sums with respect to this one sequence converges. OK? And so we've proven the claim.

Let me call the limit of this sequence something. I'm going to call it L. In fact, let's call it I for integral. Let I be limit as r goes to infinity of-- oh, I messed up my notation earlier. Sorry about that. The y's, and, x's and [? xe's ?] should have been y's and zetas. So that should have been y, zeta, y, zeta, and then y here, y here. OK? Now that's right. So I is defined to be the limit of these guys.

So now we have to do one last thing in order to prove the theorem. I have this number I, and I claim now that I satisfies the properties of the theorem-- namely, that if I take any sequence of Riemann sums-- or any sequence of tagged partitions and take the Riemann sums, that converges to I.

We've just shown that there is one that converges to. That's how I is defined. We took one partition-- sequence of tagged partitions and showed that the Riemann sums converge to some number I. So this number I depends on the partition, which I chose in the beginning. Now we want to show that this I, in fact, has that property, that no matter what partition-- sequence of partitions I take the Riemann sums converge to I. So that's the second and last thing I need to do to prove this theorem.

So claim-- let x be now any sequence of tagged partitions, which are becoming finer and finer-- so with the norm converging to 0. Then I want to show that limit as r goes to infinity of the Riemann sums corresponding to this sequence of partial sums-- [INAUDIBLE] partial sums-- this sequence of partitions exists, and equals this number I, which I obtained from this one sequence of partitions.

So once I've proven this [INAUDIBLE] once I've proven this, then I'm done. OK? This number I is therefore-- satisfies that property that, no matter what sequence of partitions I take, the sequence of Riemann sums with respect to these partitions converges to that number. Of course, here I'm calling it I, but we denoted by the integral from a, b of f of x dx.

So to prove this is not too difficult using what we have on the board. So remember, I is the limit with respect to this one sequence of partitions, and now we have an arbitrary sequence of partitions, which we want to show the Riemann sums converge to this number as well. So with y sub r-- or yr and zeta r, the partitions from before, we have-- by the triangle inequality, if I look at the Riemann sum with respect to this arbitrary partition now-- subtract I-- I want to show this convergence to 0, so I'm just going to bound this by something that converges to 0.

And then, by the usual argument with the squeeze theorem, that implies this thing converges to 0. So adding and subtracting the Riemann sum associated to the partition y and zetas-- this is less than or equal to in the triangle inequality, minus sf of [INAUDIBLE] plus now-- minus I.

And so I'm almost there. This thing now-- I use theorem three. So this is less than or equal to-- times b minus a plus-- s sub f of i sub r minus I. OK? The norm of the x sub r-- this is converging to 0. So this converges to-- since the modulus of continuity goes to 0 as the argument goes to 0, this goes to 0, plus-- remember, the same thing for this guy-- this also converges to 0, plus-- how did we define I?

It was defined as the limit of the Riemann sums corresponding to this first fixed partition. And therefore, this thing in absolute value converges to 0 as r convergence is to infinity. All right, so the absolute value of this thing, this Riemann sum, minus I is bounded by something converging to 0 as r goes to infinity. And therefore, by the squeeze theorem, we conclude that the limit as r goes to infinity of s-- so in other words, no matter what sequence of partitions we take, with norms converging to 0, the Riemann sums converge to this same number I. And that's the end of the proof.

OK. It took a lot of work to do this. This is the work of Raymond, so-- who was one of the greatest mathematicians of all time. Weierstrass was definitely one of the greatest analysts of all time. Riemann, just as a mathematician in general, was one of the greatest of all time. He not only did analysis-- he applied analysis to number theory. This is the content of the famous Riemann hypothesis, which maybe you've heard about, which is worth a million dollars-- says something about the zeros of a certain function, which then gives you information about prime numbers.

And he also had deep contributions to the foundations of geometry-- or I should say the foundations of differential geometry-- but truly some deep stuff going on. All right, so now we have this notion of a "remand integral. It's this no integral f of x, which has this property. No matter what sequence of partitions I take, with norms converging to 0, the partial sums converge to this number.

So I'm often going to denote-- so if you like, this is some alternative notation I'll use. Instead of writing f of x dx, I may just write integral a, b, f. So this definition of the Riemann integral looks terrifying if you want to actually try and compute it. And it is. So that's the miracle of the fundamental theorem of calculus is that it gives us a way to compute it.

And that's why I'm not doing any examples of computing it right now. But we can still learn about some properties of the Riemann integral without that to start. So now we're moving on to some properties of the integral. And the first is that it-- so this is a limiting process, but this limiting process is linear in f.

If I look at Riemann sums of, let's say, f plus g, that's equal to the Riemann sum of f plus the Riemann sum of g. And therefore, since integration is a limit of this-- of Riemann sums, then integration should also be linear as well. And that's the first theorem is the linearity of the Riemann sum, Riemann integration-- f and g are a, b. And alpha is in R.

So first off, alpha f plus g-- that's going to be-- so that should not be in a, b. This should be a continuous-- it's the end of a long day, OK? So sorry if I'm losing a little steam here towards the end. Anyways, alpha f plus g, if f and g are continuous-- this is a continuous function as well, so its integral is meaningful. And its integral is equal to alpha times the integral f plus the integral of g. OK?

So what's the proof? The proof is basically what I said right before I stated the theorem, that Riemann sums are linear in the function. And then we just take a limit. Let's take a sequence of tagged partitions with norms converting to 0 as r goes to infinity.

Then, if I look at the Riemann sum associated to alpha f plus g, it is easy to see from the definition, right? This is a sum from k equals 1 to [? in-- ?] so this is easy to see that this is equal to alpha times the Riemann some associated to f plus that guy. And now I just take a limit-- so take the limit as r goes to infinity.

The left-hand side converges to the Riemann integral of alpha f plus g. The limit as r goes to infinity converges to-- because we know limits respect linear operations, the limit of the right-hand side is going to be alpha times the limit plus the limit. And therefore, this is equal to alpha times integral a, b, f plus integral a, b, g.

OK. So like differentiation, it's also what one would call a linear operator. It takes a function and spits something out, but does it in a linear way. So for us, it takes a continuous function and spits out a real number in a linear way. The integral of alpha f plus g is equal to alpha times f plus g. Just like differentiation, the derivative of alpha f plus g if everything's differentiable, is equal to alpha times the derivative of f plus the derivative of g.

But Riemann integration, the Riemann integral, or integration in general, is in some sense-- I don't want to say it's not miraculous. It still is a bit of a miracle that it exists for all continuous functions, but it's not as destructive an operator as taking the derivative, if you take as your baseline continuous functions.

Last lecture we constructed a continuous function which is differentiable nowhere. So with differentiation, you can take a continuous function and not be able to take its derivative anywhere. Well, with integration, if you start off with a continuous function, you can always take it's integral. So somehow integration is a much more smoother process than differentiation. All right, so I think we'll stop there.