[SQUEAKING] [RUSTLING] [CLICKING]

CASEY RODRIGUEZ: OK. So I have to admit this is extremely awkward, lecturing to an empty room. So I have to imagine there's somebody on the other end actually listening to me at some point. Perhaps this is what YouTube stars have to go through at some point in their career.

So what is the purpose of this course? So this is for 18 100A, Real Analysis. So the purpose of this course is twofold. Really, I think the first primary purpose of this course is to gain experience with proofs. So that means being able to read a proof, being able to write a proof. And the second statement, or the second purpose, which is supposed to be a way to obtain the first purpose, is to prove statements about real numbers, functions, and limits.

OK. So the second part, this is the analysis part, OK? So for the first few lectures, we're going to do what maybe to some will be kind of review. And for most of you, a lot of this material in the first few lectures will be review. But it's a nice way to ease into the material. And things will most definitely pick up after a few lectures.

So the first set of objects we're going to define and try to prove some statements about are sets. So definition-- and because I use a lot of shorthand, I will mostly write Dfn from now on instead of the entire word, definition. So a set is a collection of objects called elements or members, OK?

Now, this course is supposed to be probably the first really rigorous course in math that many of you will deal with. So essentially, everything that we talk about will be rigorously and unambiguously defined. But we do have to start somewhere.

And so maybe you think this word, "collection," is a little ambiguous. And perhaps you should. But to actually build up set theory from the ground up would take us quite beyond the scope of this class and too far afield of the things that we want to do, or at least that what I want to do. OK, so a set is just a collection of objects called elements, or members.

There is the simplest set to define: the empty set is the set with no elements. And we denote it by this symbol here-- a circle with a dash through it. OK, so with new math typically comes new notation, new symbols that you use. So let me introduce a few shorthand notations we'll use throughout the course.

I mean, quite honestly, this is a little bit of the fun of doing higher math. You get all these funny symbols. And a very accomplished mathematician at the University of Chicago, one time said, you're really only interested in the math where that has the symbols you like to write over and over again.

So some notation-- a and this symbol which this symbol, which looks like a e-S means a is an element of S. A with a dash through this little e means everything here but. So a is not an element of S.

This upside down A means for all. It's shorthand for all. Backwards E means there exists. And a couple of more-- if you see an arrow like this, this means implies. So I've written down one thing. This implies the next statement. I'll put an arrow between them. And an arrow going both ways means if and only if, meaning if I have a statement P if and only if Q, that means statement P implies Q and statement Q implies P, all right? And if you need a quick refresher on basic logic, you can find that in the appendix of the textbook.

OK, so that's the basic definition of set, empty set. So set A-- another definition to set A is subset of B, which we write A, this little symbol that looks like a C, B If every element in A is an element in B. Little a's in capital A means little a is in capital B.

So two sets are equal-- we write A equals B-- if A is a subset of B and B is a subset of A. And A is a proper subset of B if A is a subset of B and A does not equal B. And we typically write that by A and with a dash going through a line underneath the C to signify that it's not equal. So think of it as not, so less than or equal to, but not equal to is one way to think about it.

OK, so let me say something since I'm now 1, 2, 3 definitions in. So definitions are a fact of life when it comes to math. In the beginning of any subject, there's going to be a lot of definitions because we have to have objects we want to talk about. And we have to have these unambiguously defined objects.

So it may seem like there's going to be a lot of definitions now, but this will let up. And we will start proving some theorems, which are facts, about these objects. These are the things that we're really after. We're not really after just making up definitions. Definitions are meant to be a rigorous way of defining an object we're interested in studying. We're interested in proving theorems, facts about them.

So again, a lot of this is just probably review. When we describe sets, we will use these braces and maybe list the elements in here. Or we will describe it as x in some set A, satisfying some property P of x. Or we won't write this x and A part. We'll just write all objects x satisfying x, as being an element of whatever universe we're in, that satisfy property P of x, OK?

So again, you should read this as all satisfying property P of x. So the basic examples-- and this is-- you should expect this after seeing any non-trivial definition. If you were here, I would ask you to call me out, so I'll have to police myself. But after every semi-interesting definition, you should see examples, OK? This is how you actually learn about these things, or at least digest what these things are.

So we have the natural numbers, which everyone is familiar with since they started to count-- 1, 2, 3, 4, and so on. We have the integers, which is 0, 1, minus 1, 2, minus 2. So all the natural numbers, along with their additive inverses, along with the 0 element, an additive identity, we have the rational numbers. So this is written as m over n such that m and n are integers and n does not equal to 0.

And we have R, the real numbers, which I, as of right now, cannot actually write down what they are in terms of set-building notation. In fact, this will be our first goal of the course, is to give a proper description or definition of what R actually is. But you can think of this as you did in calculus, as Q along with-- so rationals and irrationals, like pi and 2 and these things. So this is fine to think about for now.

So of course, I didn't have to use these. Maybe I'm interested in odd numbers. That's a set of numbers of the form 2m minus 1, where m is a natural number. So this is just 1, 3, 5, and so on, OK? And so note that we have the inclusions. Natural numbers are contained in the integers, which are contained in the rational numbers, which are contained in the real numbers, OK?

And if you look at the history of why these things were thought up in the first place, I mean, they were thought up to solve polynomial equations that you couldn't solve in the number system before. Integers were created because I could not solve the equation x plus 1 equals 0 in the natural numbers. Rationals were thought of because I could not solve the equation 2x plus 1 equals 0 in the integers. And the real numbers were thought of because I cannot solve the equation x squared minus 2 equals 0 in the rational numbers.

Now, I can't solve the equation x squared plus 1 equals 0 in the real numbers, which led to the creation of complex numbers. But we will not deal with complex numbers in this class. Although hopefully, if you keep studying analysis, you go on to complex analysis, which is really a beautiful subject of study to this day.

So as I said-- let me write this here-- our first goal, real goal of the class-- and this is something to keep in mind. We're not going to do it right now. Our first real goal is to describe what R is, OK? I mean, if we're going to be proving statements about the real numbers, functions of real numbers, and limits, whatever-- those limits that you learned in calculus-- then we have to be able to really describe what we're starting with, the real numbers.

OK, so let's get back to sets, to our review of sets. So there were some examples. We have a few more definitions. The union of two sets, A and B, is the set which we write-- so this is how we denote it, A U B. This is the set of all elements x. x is in A or x is in B.

The intersection of A and B-- so this was defining the union. This was defining the intersection-- is the set A cap B. And this is a set of all x's so that x is in A and x is in B. So the union is take all the things from A, take all the things from B, and put them together in one big basket. The intersection is just take the things that A and B have in common.

The set difference of A with respect to B is the set A backslash B. This is the set of all elements in A such that x is not in B. The complement of A is the set A-- so this is how I'm denoting the set. The next part is how I'm defining the set. This is a set of all elements in our universe that is not in A.

And when I say universe, I don't mean this universe necessarily. I mean, if we're looking at subsets of R, the complement is generally with respect to R. Or if all of our sets are subsets of Q, then our universe would be Q, the rationals. And we're taking the complement in there.

Two sets are disjoint if their intersection is empty, OK? So it took me quite a long time to figure out this compliment has an E in the middle as opposed to an I, as in the compliment you would give a friend. I had to do a lot of spell-checking in my thesis when my advisor pointed that out. So this is just something to keep in mind. This complement has an E in the middle of it.

OK, so let me just draw a quick picture. So this blob over here is A. This is a set B. This is a set C. In fact, let's make this a little more-- OK, let's keep C there. Then what I have here, that's A intersect B. This bit over here, with the lines going this way but not including this, this is A take away B, A backslash B.

And OK, so that was not meant to be along the same direction as this one, so let's go vertical. And everything with a vertical line is A intersect B, OK? So A backslash has the lines going this direction. A intersect has the lines going this direction. A union B has the lines going vertical. And C is way over here not touching any of A and B. So A and C are disjoint and B and C are disjoint. OK, they have nothing in common.

OK, so this was a lot of definitions. We have not proven a single statement yet, so it's about time we do. This is probably one of the most basic theorems one can prove at the start of a Real Analysis class or any class about proofs. This is analogous to when you write your first Hello World program in a programming class.

So let me state the theorem, which is DeMorgan's Laws. And the statement is the following. So if A, B, C are sets, then I have several things I can say. The union of B and C, taking their complement, this is the intersection of the complements. So the complement of the union is the intersection of the compliments. If I take their intersection and take the complement, this is the union of the complements. So the complement of the intersection is the union of the complements.

Now, these are complements, meaning I am, in some sense, taking a set difference with respect to the entire universe. But I can make these things relative to some set A. So A take away B union C, this is the same as A take away B intersect A take away C. Really, again, you should think of this as a special case of one. Or at least if you were to write the proof-- I'm not going to because it's all going to be contained in the first two-- then you would see it's really just a proof of this guy. A take away B intersects C equals A take away B union A take away C, OK?

So again, for a quick refresher about logic I would look at the appendix of the textbook. In general, so let me make a few remarks before we move on to the proof about typically how this is going to look. So this is some remarks.

Typically, a theorem is a statement of the type P implies Q. Let me write this out in English. If some statement P holds, then Q-- for us, it's if I have any three sets, then I have these equalities between these operations of sets. So the general structure you'll see of the class is I have objects which I define unambiguously. I want to prove theorems now, meaning true statements about these objects. And the real meat is the proof part.

So what is in this mysterious guy, the proof? It's quite simple. You start with-- you assume P, meaning what you were given, the hypotheses, the hypothesis, P, and-- I'm going to put dots here-- through logic and most definitely, most of the time some calculations, you arrive at Q is true. And most proofs are ended with this little box here, OK?

So most proofs have this structure. I take my hypotheses. And these hypotheses mean something in terms of the definitions I have given. And now, I need to use these unambiguous definitions, along with logic and maybe some calculations, to conclude that statement Q is true. That is the essence of a proof. That is all there is to it. Now, that doesn't mean it's a simple thing to learn how to do. That's the point of this course. But distilled down, that's what a proof is, OK?

And Q-- so I said P usually means something in terms of the definitions we have. But also, Q will usually mean something in the definitions that we've given. And so our job is to verify Q. So let's go with proving this theorem. And in fact, I'm only going to prove property 1. Property 2, 3, and 4 I'll likely put on the homework.

So let B and C be sets. So I mean, this is the only hypothesis I get. I'm trying to prove that B union C complement equals the intersection of the complements. So what does that mean? So we want to prove. So this is-- it's quite helpful, especially when you're first starting to do proofs, to write down what you're actually trying to prove.

So even though I have this statement here, it's an equality between two sets. Equality between two sets means something specifically, right? We have that in our definition-- where is it-- over there that two sets are equal if one is a subset of the other and vice versa. So that's what we have to prove. We have to prove that the left side, B union C complement, is a subset of B complement intersect C complement and vice versa.

So we want to prove that is a subset of B and-- OK? So that's what the equality means. That's what we have to prove. We have to prove those two statements now, OK? And that's as distilled down as far as we can go.

So let's prove this. Now, we'll prove this using, again, logic and what these things actually are. So let's prove this first statement here.

So I have to show that every element in this set is an element of this set. So I'll even write this down as WTS. That means Want To Show. This is the first thing we'll show. As we go on, I'm not going to write as much as I'm doing right now. But this is the first theorem and proof you're seeing, so I should write down quite a bit.

So the first thing we want to show is we have this inclusion, OK? That means every element here is an element here. So let x be in B union C complement. And now, we're going to trace what this means. And we'll eventually arrive at x as in this. So then x is not in B union C. That's just the definition of the complement.

Now, x is not in B union C means x is not in B and x is not in C because the union is-- something's in the union if it's in B or C. So something's not in the union if it's not in B and not in C. Now, this implies, simply again by the definition of what it means to be in the complement, x is in B complement and x is in C complement.

But this, again, is simply the definition of x being in B complement intersect C complement, OK? So you see, we started off with an element in this guy and we showed that it's also an element of the right-hand side. So thus, B union C complement is contained in B complement intersect C complement. Now, we want to do this other inclusion here.

Now, this is one of those rare situations where you get to essentially reverse the entire argument and get what you want. But let's just go through it in a linear fashion. Let's take something from here and show it's in here. So let x be in the intersection of the complements. Then that means x is in B complement and x is in C complement. That means x is not in B-- so that's this statement. That's the definition of being in the complement-- and x is not in C. That's, again, the definition of being in the complement.

Now, just like we used here in this step, this is equivalent to-- so really, I should-- in this statement, I should have written this statement is equivalent to this statement, but we'll remove that. So x is not in B and x is not in C. This means x is not in their union, which implies that x is in the complement of the union, OK?

So thus, we've proven is a subset of B. And since we've shown both sets are a subset of each other, that means, by the definition of two sets being equal, they are equal. Again, this box means really nothing. It just means that's the end of the proof.

All right, let's move over here. This is terrible. And not everybody uses that little box to finish a proof. Some people don't put anything. When I was in graduate school, I was a TA for this guy named Paul Sally who was a fantastic teacher and really loved math, who would end--

So amazing story about this guy is, when I was his TA, he was in his 70s, I think. But he had also had diabetes. So he had lost both of his legs beneath his knees. He was also legally blind. And he had a patch over one eye. So he himself often referred to himself as the a pirate mathematician. But he would end his proofs with-- at least in his textbook-- he didn't ask me to do this on the board, thankfully. He would end his proofs with a picture of himself with this cob pipe that he had, very much in the pirate fashion.

Anyways, OK, moving on from things that end proofs, let's go on to a next subject, induction. So induction is a way to prove theorems about natural numbers, OK? The theorem itself is more of a tool rather than an interesting fact on its own, OK? So let me state the theorem. And then we'll go over a couple of examples on how to use induction.

So let me recall from-- I think I just erased it. N is the natural numbers. And it has an ordering, meaning-- so we'll precisely define what ordering means. But just in your head, this means the usual thing-- 1 is less than 2 is less than 3 is less than 4.

So a property of the natural numbers, which will take as an axiom, is the well-ordering property. So an axiom is not something you prove. You assume this about the objects that you've defined or are studying up to this point.

And so the statement is if I take a subset of natural numbers, which is non-empty, then S has a least element or smallest element. Now, what does this mean? Let me write this last statement out. i.e. There exists an x in S-- "st" I will often write, meaning such that or so that-- such that x is less than or equal to y for all y in S, OK? So every non-empty subset of the natural numbers has a smallest element, OK? We're going to take that as an axiom, as just a property of the natural numbers, which we'll assume.

Now, using this axiom, we're going to prove-- it's not really often you hear it called as a principle of mathematical induction, but this will state it as a theorem instead of a principle, whatever a principle is supposed to be. So induction, so this is due to Pascal. Or at least in its first rigorous formulation is let Pn be a statement depending on natural number n. OK, so maybe we have some equality between two quantities that involves a natural number n, OK? That could be our statement P of n.

Now, we're going to assume-- so what are our hypotheses about this statement? What's our if? Assume that this statement satisfies two properties. This first property is usually referred to as a base case. That is that P of 1 is true. And the second property is called the inductive step.

So this statement satisfies the following property that if you assume P of m is true, then you can prove that P of m plus 1 is true. So I have a statement which satisfies both of these properties, OK? In particular, since I'm assuming P of 1 is true, by the second property, P of 2 is true. And then again by the second property, P of 3 is true. And then P of 4, and then P of 5. And so if you followed that last line of reasoning, this means you should be able to guess what the conclusion of this theorem is. Then P n is true for all natural numbers, OK?

All right, so we're going to use the well-ordering property of the natural numbers to prove this theorem about the induction. OK, so we have our assumptions. I'm not going to-- although, I said over there let B, C be sets, I'm not going to rewrite the assumptions that we have about our statement P. We're just going to start trying to prove P of n is true for all n.

So let me write our conclusion slightly differently. Let S be the set of all natural numbers such that P of n is not true. So what I want to show is that P of n is true for all n. So that's equivalent to saying we want to show that S is empty, OK? The set of natural numbers where P of n is not true, this is empty. This is equivalent to saying P of n is true for all n.

And the way we're going to do this is another staple of mathematical proofs is trying to prove this by contradiction, OK? So what does that mean? Let me make a few comments about what that means, proof by a contradiction.

OK, so in a proof by contradiction-- so this is-- what I'm about to write down is not part of the proof. This is commentary not to be included in the proof. What does it mean to say we're going to prove S is equal to the empty set by contradiction?

We're going to assume that the statement we want to prove is false. Or not false, but we want to assume that the negation of the statement we want to prove is true and then arrive at a false statement, OK? So we want to assume-- this is what we're going to do. We're going to assume the negation of the statement we want to prove-- namely, S is non-empty, OK?

And from this, we want to derive a false statement, OK? And so if we are to do-- if we were able to do that, then-- let me just say, again, you can check in the appendix or you can just believe me that the rules of logic then say that our initial assumption, that S was not empty, is false to begin with, OK?

So rules of logic, meaning I cannot start from a true assumption and derive, in a logically consistent way, a false statement, OK? That is, if we believe that the rules of logic we're using are consistent, which that's a little bit hairy to talk about. But for our purposes of our class, you can believe me that the rules of logic we use-- or at least accept that the rules of logic we're going to use are consistent and sound.

OK, so back to the proof at hand. We have this set of natural numbers where the statement is not true. We want to show it is empty. We're going to do it by contradiction, meaning we're going to assume the negation of the statement we want to prove-- namely, S is non-empty. And we're going to derive a false statement from that assumption, OK? And by the rules of logic-- that means that our initial assumption-- that S is non-empty-- is, in fact, false, OK?

All right, so towards a contradiction, suppose that S is non-empty, OK? Now, we're going to use the well-ordering property of the natural numbers. By the well-ordering property of the natural numbers, S has a least element, x, OK?

Now, what do we know about x? So first off, x cannot be 1, OK? S is a set where this property does not hold. x cannot be 1 because-- let me again rewrite this fact that S has the least element. Let me just reiterate that S has a least element in the set, OK?

Now, x cannot be 1 because we're assuming the base case, meaning P of 1 is true. So since P of 1 is true, that means 1 is not an S, which means x is not 1. In particular, x must be bigger than 1. So x is some magical natural number out there bigger than 1 that's the least element of this set S.

OK, since x is the least element of S-- so let me draw. On the number line, we have 1, 2, 3, 4. Out there is some magic point x, which is the least element of S. And the rest of the subset S lies to the right of this number x, right? Because it's the least element of S.

And therefore, x minus 1 cannot be in S. So since x is the least element of S and x minus 1 is less than x, this means that x minus 1 is not in S. Otherwise, it would be a smaller element than x in S. So thus, what does it mean to not be in S? It means that P of x minus 1 is true. By the definition of S, this means P of x minus 1 is true, OK?

But by the second property we're assuming about our statement P, this means that the next guy in line, x minus 1 plus 1, is true, which is just x, which means that x is not in S, OK? So from the assumption-- so let me just recap. From the assumption that S is non-empty, we've derived two facts. 1, x has the least element in S. And that element is also not in S.

So written out, we've concluded there exists a natural number which is both in S and not in S. And this is a false statement. You cannot have an object and member that's both in the set and not in the set, OK? And at the end of contradiction arguments, I'll usually put two arrows hitting each other. So that's a contradiction. Therefore, our initial assumption that S is non-empty has to be false. And therefore, S is the empty set, OK?

So I encourage you to go through that proof a little slowly because maybe you got turned around by taking the complements or the general scheme of how a proof by contradiction works. But don't spend too much time on it because, as I've said, this theorem itself and its proof are not the thing we're really interested in. Or at least, it's not the most interesting. It's more of a tool that we'll use to prove more interesting statements.

OK, so how do we actually use this theorem, induction, to prove other statements? So I guess I should include this here. This falls under the umbrella of logic, meaning we're going to approve previous-- we're going to use previous statements we've proven to prove new statements. But anyway, so how do we use induction in practice?

So if we want to prove some statement-- for all n, Pn is true-- in the print then, this theorem about induction-- this theorem of induction-- tells us we just have to do two things, OK? We have to prove the base case. And this is usually easy. You just stick the number 1 into the statement that you want to prove. And that's the end of the story.

And the second step is usually-- or the second thing we have to prove is the more involved part, which is we have to prove that the statement that if PM is true, then P of m plus 1 is true, OK? If we want to do a proof by induction, there's two smaller proofs we have to do.

First, we have to prove P of 1 is true. And then we have to prove this statement. If P of m, then P of m plus 1. So again, this is usually referred to as the base case. This is the inductive step. So let's try and actually do this.

All right, so-- so another question I get at the beginning of a course, especially about proofs-- because there's a lot of uncertainty about what you can assume is true, what can you use, what can you not use, right now, at this point, you can use whatever you know about any of the algebraic properties, you know about the real numbers, the rational numbers-- and by algebraic properties, I mean if a plus b equals c, then a plus b times d equals c times d-- and what you know about inequalities.

So we're going to go much more in depth into ordering, which is what inequality is a part of. But you can use all the properties you know about solving inequalities or manipulating inequalities, meaning if I have one number is less than or equal to another number, then when I multiply both sides by a positive number, that doesn't change the inequality. So you can use all of these algebraic properties of rationals and real numbers from here on out. I mean, so we're going to be proving things about calculus. So you certainly cannot use anything about continuity, differentiation, or anything like that. But for now, you can use all the algebraic properties you know.

So the first statement we're going to try and prove using induction is the statement that for all c not equal to 1, for all n, a natural number, 1 plus c plus c squared plus c to the n equals 1 minus c to the n plus 1 over 1 minus c, OK? So this here is our statement P of n. It depends on the natural number n, OK?

So we're going to do this by induction, which means we're going to do those two things. We're going to prove the base case, P of 1, which I said is easy. And then we're going to prove the second case, the second property, the inductive step, which is a little more involved, but not so much involved, at least in the beginning.

So let me call this inequality star. We're going to prove star by induction. So first, we will do the base case. And like I said, the base case is usually you just plug in n equals 1 and verify that P of n is true. And that's what we do.

1 plus c to the 1, which is the left-hand side, does, in fact, equal 1 minus c to the 1 plus 1 over 1 minus c because this right-hand side-- 1 minus c squared is 1 minus c times 1 plus c-- the 1 minus c's cancel. And so the base case is proven, all right?

Now, we do the inductive step, OK? So we're going to assume that star holds for n equals m So we're going to assume P of m. So assume that 1 plus c plus c to the m equals 1 minus c over 1 minus c, OK? Now, we want to show-- again, let's write out what we want to do, what's our plan. We want to prove that this equality, that this line star holds for n equals m plus 1, OK?

So again, what I wrote here, this is basically star for n equals m, OK? And let me call this second inequality-- the second equality 2 star. So this is my assumption for m, n equals m.

OK, so let's take the left side for n equals m plus 1 and see if we cannot massage it to get the right-hand side for-- I should say the right-hand side for n equals m plus 1. So here is the calculation part. So we have 1 plus c plus c to the m plus c to the m plus 1. This is the m plus 1 case of the left-hand side of star, which we want to show is to the-- is equal to the n equals n plus 1 case of the right-hand side.

Now, this is equal to-- now, we already know what this is equal to by our assumption. This is by the second star there, which is what we're assuming is true. This is equal to 1 minus c n plus 1 over 1 minus c plus cm plus 1. And so now, we just do a little bit of algebra. This is equal to 1 over 1 minus c m plus 1 plus c m plus 1 minus c m plus 2 over 1 minus c. Those cancel and I'm left with 1 minus c to the m plus 2. And I'll write it just so that you can see this is really the m plus 1 case, all right?

So again, we arrived at this first step by our assumption, the second starred equation, OK? So thus, star holds for n equals m plus 1. So by induction-- or really, I should say the theorem of induction that we proved-- our equality between those two objects, or two expressions, is valid for all n, OK? OK.

OK, so let's do one more example of using induction. So let's prove if C is a real number bigger than or equal to minus 1, then for all n, a natural number, 1 plus c to the n is bigger than or equal to 1 plus n times c, OK?

All right, so we're going to do this by induction again. That means we need to prove the base case and we need to do the inductive step. So base case, as always, will-- so this is just right here. We're going to do this by induction. So as you can see, the base case is, again, n equals 1 is clear just by looking at it. 1 plus c to the 1, in fact, equals 1 plus 1 times c. So it's certainly bigger than or equal to 1 plus n times c.

So I think that's the last stars I'll use for this lecture. So our statement, our inequality star star star, holds for n equals 1. All right, so that's our base case. Now, we're going to assume that this inequality holds for n equals m and try to prove that it holds for n equals m plus 1.

So we're assuming this when n equals m. So 1 plus c to the m is bigger than or equal to 1 plus m times c. And we want to prove this inequality with n equal to m plus 1. And we're just assuming this guy, OK?

So I want to get the statement for n equals m plus 1. One way to do that is this left side. I want to get-- let's look at the n equals m plus 1 side and see what we can do with it. So again, this is a calculation part and logic. So we have 1 plus c to the m plus 1. So that's the n equals m plus 1 side of this.

This is equal to 1 plus c times 1 plus c to the m. Now, we're assuming, again, this inequality. This is the n equals m case. So we can use it. So we're assuming it. We use it. And since C is bigger than or equal to minus 1, 1 plus C is non-negative. So this thing is bigger than or equal to this thing. So if I multiply both sides by 1 plus c, I preserve the inequality. So this is bigger than or equal to 1 plus mc, OK?

Again, this just follows from essentially the assumption multiplied through by 1 plus c, OK? So now, I'm going to finagle this. So let me just-- I'm not doing anything different here. I'm just going to rewrite this over here so that I can have a chain of inequality.

So I have 1 plus c to the m plus 1 is greater than or equal to 1 plus c 1 plus mc. All right, so now, this is bigger than or equal to this. And this here-- so when I write equal, I do not mean that this left side is now equal to what I'm about to write here. That means the previous thing on here is equal to what I'm about to write here, OK/ this is a typical fashion and writing down inequalities-- or I guess, practice.

So this is equal to 1-- so just doing the algebra-- m plus 1 times c plus m times c squared, OK? Now, this part is exactly the n equals m plus 1 side of this. And I have a little room to give because now this is plus something that's non-negative.

So let me just rewrite this again. This means that 1 plus c to the m plus 1 is greater than or equal to 1 plus-- so again, I'm kind of writing a lot here. I will stop writing as much as the course goes on. But I encourage you, especially in the beginning, to write all the steps and logic, OK? So again, I'm not rewriting anything. I'm just summarizing what I've done here.

Now, this right-hand side-- so I have this as bigger than or equal to this. And this right-hand side, since I have a number plus something non-negative, m times c squared-- m's a natural number. This is bigger than or equal to 1 m plus 1 times c. Thus, 1 plus c to the m plus 1 is greater than or equal to, which is the n equals n plus 1 case. So by induction, this inequality triple star holds for all n. All right.