[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIGUEZ: So let's continue our discussion of series. So last time, we proved the comparison test at the end of last time. So this was the comparison test. So which is a statement about series with non-negative terms, with one being smaller than the other.

And one of two things, or two things are true. If I have xn and yn non-negative with yn bigger than or equal to xn, then the conclusion is, if the bigger series converges, this implies that the smaller series converges. And the second statement is, if the smaller series diverges, then the larger series diverges.

And we also proved not using the comparison test, but for p-series, that the series 1 sum from n equals 1 to infinity of 1 over n to the p, This converges. Well, I mean, we did use a comparison test for one direction, I guess. So 1 over n to the p converges if and only if p is bigger than 1. So this converging implied p has to be bigger than 1 by the comparison test and what we know about the harmonic series 1 over n, sum of 1 over n.

So typically, in applications you use these two theorems together. To say something about series that don't look so simple. So for example, if I look at the series, 1 over n squared plus 2020n, n equals 1 to infinity. So one could ask, does this converge, diverge?

Well, this is a series with non-negative terms. And remember, I mean, the bigger thing has to converge to imply that the smaller thing converges. And if the smaller thing diverges, then the bigger thing diverges. So don't get the inequalities mixed up. So I have 1 over n squared plus 2020n.

This 2020 times n is just making things bigger on the bottom and therefore smaller overall. So this is less than or equal to 1 over n squared. And since this converges, this implies by the comparison test that sum from n equals 1 to infinity of 1 over n squared plus 2020n also converges.

Now, a mistake I'm sure we've all made at some point is mixing up the inequality and not getting quite the right answer. So for example, it is also true that 1 over n squared plus 2020n is certainly less than or equal to 1 over 2020 times n, which is also less than or equal to 1 over n.

And so, you're tempted to say, since this diverges, it implies the other series diverges. But this is not right. Because the inequality is wrong. Remember, if you want to apply the comparison test, you either have to have a bigger series which converges or a smaller series which diverges.

Here we came up with a bigger series which diverges, which gives us no information at all. If we have a bigger series, like 1 over n squared that we know converges, then do we get information about the original series.

So let's do another well-known test, or at least test you should remember from calculus, so-called ratio test. So what is a statement of this? So suppose xn does not equal 0 for all n. And this limit l equals limit as n goes to infinity of x to the n plus 1 over x to the n, an absolute value exists.

Then, if l is less than 1 this implies that the series xn converges absolutely. And if l is bigger than 1, then the series diverges. Now, what about l equals 1? There's no information for l equals 1. Meaning you could have a series so that this l equals 1.

And you could have a series where it diverges. And you could also have a series where l equals 1 and the series converges. So for example, if xn equals 1 over n, for example. Or I mean, even worse, let's say 1 for all n. But we know that the series 1 diverges. Because the individual terms do not converge to 0. They're just 1 the whole time.

But for xn equals, let's say, 1 over n squared, so the one we just saw a minute ago, implies l, which is the limit as n goes to infinity of n plus 1 squared over n squared equals the limit as n goes to infinity. Dividing through, one plus 1 over n squared, this equals 1 plus 0 squared equals 1.

So for this series, the root or the ratio gives this l is 1. And this series converges. So again, for the case that l equals 1, we have no information. We can't say anything based on-- this theorem gives us no information.

So we're going to prove this theorem basically by-- so we're not exactly going to use a comparison test. But we are still going to compare this series satisfying one of these two assumptions to a series we know, which is namely the ratio test for number one. So first off, let's get number two out of the way.

So let's suppose l is bigger than 1. Let alpha be a number between 1 and l. Actually, we don't even need to do that. So here's-- here is 1, here is l. So since xn plus 1 over xn absolute value converges to l, there exists an integer M0.

So that's a natural number. So that for all n bigger than or equal to M0, xn plus 1 over xn in absolute value is bigger than or equal to-- so here, you could write 1 as-- this is equal to l plus-- I should say, minus-- l minus 1. So think of this as being epsilon in the definition of convergence. And so, this is over here. l plus l minus 1.

So by the definition of convergence, for all n sufficiently large, xn plus 1 over xn in absolute value has to be in this interval. So it has to be bigger than or equal to 1. Which implies that for all n bigger than or equal to M0, xn plus 1 is bigger than or equal to xn.

And so, I could write this as saying that x M0 is less than or equal to x M0 plus 1, is less than or equal to x M0 plus 2, and so on. But this implies that these xn's cannot converge to 0 as n goes to infinity. Because for all n bigger than or equal to M0, they are increasing. And the only way for an increasing sequence which is non-negative to converge to zeros for them all to be 0. And we're assuming they're not 0.

So that proves two. Let's prove one. So suppose l is now less than 1. And now, let alpha be a number between l and 1. So I just want to give myself a little bit of room, to work as you'll see in just a second. So by the same reasoning as before, so here's l, alpha 1 alpha. I can write as l plus alpha minus l.

So I should think of this as kind of an epsilon. Then since xn plus 1 over xn in absolute value converges to l, which is less than alpha, there exists in M0 natural number. So that, for all M bigger than or equal to M0, xn plus 1 over xn in absolute value is less than or equal to alpha.

So here I have alpha just a little bit bigger than l. And if I draw-- so this is l minus alpha minus l. So since I have this sequence here converging to l for all insufficiently large, this should be in this interval. Meaning this should be less than or equal to alpha.

Then for all n bigger than or equal to 0, so let me just write this slightly differently, x sub n plus 1 is less than or equal to alpha times the absolute value of x of n. Now, let's see what this means. For all n bigger than or equal M sub 0. Let's say I look at the absolute value of x sub n.

So now, this n has nothing to do with these n. It's just-- so then the absolute value of x of n is less than or equal to x-- so maybe I'm off by-- OK, so let's increase that by 1. Well, so let's not be confusing here. Let's make this l. Let's make that l. So for all l bigger than or equal to M0 plus 1.

So think of l as playing the role of being n plus 1. This is less than or equal to x to the l minus 1, which is less than or equal to x to the l minus 2, now with a square. So what I have here is now alpha times alpha x to the l minus 2.

And now, let's drop again by 1. And I can do this as long as this quantity here is bigger than or equal to M. So I get this is, if I keep doing that, this is less than or equal to alpha to the l minus M0, x to the M0.

And so, actually let's, we shifted things-- yeah.

So now we're going to use this to bound the partial sums of the x of n. So let M be a natural number. And if we look at partial sum, sum from n equals 1 to M of x sub n, this is equal to sum n equals 1. to M0 xn plus sum from n equals M0 plus 1 to M. x sub-- now let me just change dummy variables for signing, for indexing this.

So this is equal to the sum from n equals 1 to M0 of xn. And now, this, I'm going to use this inequality here. So this is kind of a mess. So this is the inequality that I get from this. Remember, alpha's less than 1. Plus-- OK. So time's l equals M0 plus 1 to M. OK, everything's fine. Times alpha to the l minus M0 plus 1.

So all I did here was replace this absolute value of x to of x sub l. l is just a dummy variable for indexing these guys, but using this inequality here. So, sorry, I kind of messed that up a little bit. But the important thing is that we have this inequality, which somehow tells you this series is not that far from being a geometric series, at least when l is big enough.

So then, when we take an arbitrary partial sum, we split it up into two parts. The stuff that comes up to this integer sub 0, which we don't care about, that's just a fixed number, plus this part that we care about, which is after this fixed number M0. And we use this inequality to replace this by x sub M0 plus 1, alpha to the l minus M0 plus 1.

The thing to remember is that little m is the thing that's changing. So we're trying to bound this independent of little m. Capital M sub 0, that's just something fixed. That could be 1,000. So this is equal to sum from n equals 1 to M0 xn, plus x sub M0 plus 1.

And now, if I change the variable again, so l starts at M0 plus 1 and ends at M. So this is now a sum if I go-- so now, for the second sum, n is equal to l minus M0 plus 1. So now, n starts at 0 and ends at M minus M0 plus 1, alpha to the n. Again, Emma, is the little-- is the thing that's changing here. And we're trying to bound this independently of little m.

But now we're in good shape. Because this looks like a geometric series. Alpha, remember, is less than 1. So this is less than or equal to sum from n equals 1 to M0 x sub n plus x sub n 0 plus 1 times-- now, instead of just being a sum from 0 up to M minus M0 plus 1, why not throw all of it in there. And this is equal to n equals 0, xn, plus x to M0 plus 1 times 1 over 1 minus alpha.

Remember, alpha is a number that we fixed to be less than 1. If you try to do what we did before and not fix the alpha just a little bit to the left of 1, and try to do everything with 1, you would have wound up with 1 to the n here. And that wouldn't have finished the proof. That wouldn't have closed the proof. But given yourself a little bit of room, which is why we fix this alpha.

Now, this number here is independent of little m. That's the whole point. So what have we proven? That for all natural numbers little m, the nth partial sum is less than or equal to a fixed number given by sum from-- OK. And therefore, this sequence of partial sums is bounded and therefore converges.

So kind of the simplest application of this is maybe a series that looks familiar. So for all x-- so this is I guess a theorem slash example. But for all x in R, the series x to the n over n factorial, n equals 0 to infinity converges absolutely here. 0 factorial is, of course, 1.

And you just use the ratio test. So of dx to the n plus 1, or the n plus first term over x to the n. So absolute value, and this equals the limit as n goes to infinity of n plus 1 factorial is equal to n plus 1 times n factorial.

So this cancels with that n factorial. And I get x over n plus 1, limit as n goes to infinity of just this fixed number over n plus 1 equals 0. And this is certainly less than 1. And therefore, by the ratio test, this series converges absolutely.

So I got a little hung up on exactly the indexes and matching them up precisely. But the important thing to take home from this proof was that when this ratio is less than 1, then this series behaves very much like a geometric series for as the terms get very-- as you go far enough out in the terms.

And this idea of just trying to relate the series to a simple series that you know a lot about, I mean, basically the only series that you know everything about, namely even how to sum it, this is how you get this test. And this is also even simpler how you get the next test, which is the root test.

So the root test, let's take a series. And suppose this l, this limit l equals the limit as n goes to infinity of x sub entity 1 over n exists. Then, two conclusions, just kind of just like in ratio test. If l is less than 1, and this implies that the series converges absolutely. And if l is bigger than 1, then this series converges-- no, the series diverges.

And again, just like in the ratio test, no information for l equals 1. You take the same series that we looked at before. Did I leave it up there? Yes. So xn-- x sub n equals 1 for all n. This limit here, l, exists. It's equal to 1. And that series diverges. If you look at 1 over n squared, and I take this limit, I again get 1. But that series converges. So for capital l equals 1, we get no information.

So let's-- and in this case, it's even clearer how we're relating the series to a geometric series. So let's-- in my notes, I proved two first both times. So perhaps I should have written it one, two. I'll know for the future. So suppose l is bigger than 1. And now, we're going to show that this series diverges, again, by showing that the terms do not converge to 0.

So it's the same idea as before. Here's l, here's 1. So for all n sufficiently large, x to the n to the 1 over n has to be inside this interval.

And since x to the n, x sub n to the 1 over n converges to l, which is bigger than 1, this implies there exists an integer in 0 such that for all n bigger than or equal to M0, x sub n to the 1 over n is bigger than 1, which implies all n bigger than or equal to M0, x sub n in absolute value is bigger than by just taking powers of both sides is bigger than 1.

Now, since all of the absolute values of x sub n's is bigger than 1, this implies that this cannot converge a 0. So remember, I mean, let me remove this absolute value. x sub n cannot converge to 0. Why? I mean, we could go back to the basic definition of convergence, of what it means for a sequence to converge to a real number x, and what it doesn't mean, or what it means for it to not converge to x.

So xn does not converge to 0 if there exists a bad epsilon. So that x sub n is outside of that interval. As long as is outside of that interval if I go far enough out. And we certainly have that here.

Or you could use-- so this bigger than 1, in fact, implies that the lim sup of x of n is bigger than 1. Because if I have two sequences, one bigger than the other, then the lim sup of the bigger one is bigger than or equal to the lim sup of the other one, which is 1.

And you're doing in this week's assignment, that this converges-- that xn converges to 0 if and only if the lim sup of the absolute values of xn's converge to 0. And this is not. It's bigger than or equal to 1. So we'll leave that there.

So now, for the other case, that l is less than 1. Suppose l is less than 1, then let alpha be a number between l and 1. Again, since this converges to l, which is less than alpha, for all n sufficiently large, this has to be less than alpha.

So there exists an integer, I have x to the n to the 1 over n is less than alpha, which implies that for all n bigger than or equal to M0, x to the n in absolute value is less than alpha to the n.

So here again, we're seeing this series, if it satisfies these hypotheses, is very much acting like a geometric series. Remember, alpha's less than 1. Then for every natural number little m, if we look at the nth partial sum of the absolute value, let's say n equals 1. This is we split it up again into a part that we don't really care about, plus an interesting part.

Remember, little m is the thing that's changing. This should look like a little m compared to capital M0. And this is less than or equal to sum from n equals 1 to M0, x sub n. Again, this is just a fixed number. Plus now we can put this inequality, n equals M0 plus 1, m alpha to the n. And this is less than or equal to sum from n equals M0.

I'm going to go kind of fast here, because I'm running out of space on the board here. Maybe I'll write this out in just a minute. But why this is true. And this part here is less than or equal to 1 over 1 minus alpha. Because what do I do? This sum is a finite sum. And it's certainly bigger than if I make the lower bound smaller and the upper bound larger.

And this is just the nth partial sum corresponding to a geometric series with non-negative terms, the alpha to the n. And so, that's less than or equal to sum from n equals 0 to infinity of alpha to the n, which equals 1 over 1 minus alpha. So that's how we got this term.

And therefore, the partial sums corresponding to the series with absolute values is bounded. And therefore, that series with the absolute values converges. And we have absolute convergence.

So now, let me state a theorem about alternating series. It is not-- I prefer not to call it an alternating series test, because there's nothing really to test. I mean, at least with the ratio and root test, you have to recompute a limit, which might require some work to do.

And therefore, at least to me, that's kind of a real thing, that you have to do a little work to test whether a series converges. And for alternating series, the test is, you look at it. And that's it. You don't compute anything. You look at it. So I prefer not to call this theorem about alternating series an alternating series test.

So the theorem is, for alternating series. And the statement is the following. Let x sub n be a monotone decreasing sequence converging to 0. So because this thing is monotone, and monotone decreasing and converging to 0, it's all-- so let me-- I'll put this in parentheses.

Therefore xn is bigger than or equal to 0 for all n. I cannot have a monotone decreasing sequence converging to 0 if one of the xn's is less than 0, because they keep getting smaller.

This is it. This is not-- this is all of the hypotheses. You don't have to compute anything. Then the series minus 1 to the n, xn, n equals-- let's of course, we don't have to start at 1 in particular. But at least for this statement, let's make it precise. Sum from n equals 1 to infinity of minus 1 to the n, x to the n-- x sub n converges.

And we can just say convergence, not necessarily absolute convergence. Because again, if we look at minus 1 to the n, 1 over n, which is 1 over n is a monotone decreasing sequence converging to 0. That converges, but not absolutely. And if we have x sub n equal 1 over n squared, again, 1 over n squared is a monotone decreasing sequence converging to 0. That would converge absolutely. So we just have a statement about convergence.

So how we're going to do this is, we are-- it's kind of like how we proved convergence for p series, in that we're going to show that a certain subsequence of partial sums converges. And then, we're going to use that to show that the full sequence of partial sums converge.

So let me state this as claim one. So the subsequence of partial sums, S sub 2k, so this is just the sum from n equals 1 to 2k. We're going to show this converges.

So again, just to be complete, S sub m, this is the nth partial sum. it is the sum from m equals 1 to m. So how we're going to do this is, we're going to show that these partial sums are in fact monotone decreasing and bounded from below. Monotone decreasing, basically because these guys are monotone decreasing. And bounded from below by the same reasoning.

So let's show that. For k, a natural number, if I look at S 2k, so this is a sum. It equals 1 2k, minus 1 to the n x sub n. Now I have the n equals 1 term, which I can write as minus x-- so just writing this out in a certain way.

So this is minus x1 plus x2 minus x3 plus-- and then n equals 2k is even. So then I get plus x 2k. This is equal to x sub 2 minus x sub 1, plus x sub 4 minus x sub 3, plus all the way to x sub 2k minus x sub 2k, minus 1.

Now, this sequence is monotone decreasing. So x sub 2 going to be smaller than x sub 1. x sub 4 is going to be less than x sub 3. x sub 2k minus x sub 2k minus 1 is less than 0. So since this is monotone decreasing, this is bigger than or equal to. So what I just said has nothing to do with this inequality I'm writing here-- yet.

So I kept everything here. And now I add x sub 2k plus 2 minus x sub 2k plus 1. So what I was saying was that the x sub n's are monotone decreasing. This term is right after this term. So it's smaller than this term. So this term here is less than or equal to 0. And therefore, this thing plus itself plus something that's negative is less than or equal to this thing.

Now, what is this thing on the right hand side? This is just a x 2k minus plus x 2k, minus x 2k plus 1, plus x sub 2, k plus 1. And this is the k plus 1 2k plus 1 partial sum. Thus this series is monotone decreasing.

So to show that the subsequence of partial sums converges, we just have to show that this subsequence is bounded, or bounded below. If it's monotone decreasing, it's already bounded above by something. So we just need to show it's bounded from below.

Now, for all k, a natural number, let's look at S sub 2k and group term slightly differently. So this is equal now to-- I'm going to write it as minus x1 plus x sub 3. x sub 2 minus x sub 3, plus x sub 4 minus x sub 5, plus so on, plus x 2k minus x 2k minus 1. OK, I think-- ah, no, that's not right.

So this is minus 2 plus x of 2k. All right. Now, this is right. So this is equal to minus-- so again, the x sub n's are monotone decreasing. So x sub 3 is smaller than x sub 2. So this is non-negative. This is also non-negative, because x sub 5 is smaller than x sub 4, so on and so on.

So this is also non-negative. So S sub 2k is bigger than or equal to minus x sub 1. Plus non-negative terms, so 2k. And remember, x sub n's, these are monotone decreasing converging to 0. And therefore, that they're all non-negative. So this is bigger than or equal to minus x sub 1.

Thus, what do we get? So for all natural numbers k, I have minus x1 is less than or equal to S sub 2k. And because these are monotone decreasing, this is always less than or equal to S k equals 1.

And this is-- so if you like S sub 2, is just minus x sub 1 plus x sub 2. So this sequence of numbers is bounded between this real number and this real number. And therefore, it's bounded. And since it's monotone decreasing, it has to have a limit. So S sub 2k converges.

Let's call this limit something. Let's call it S. And we'll show that the sequence of partial sums converge to S now. We just showed this for along a sequence of-- along a subsequence of partial sums. Let's show it converges to S along the full sequence.

So this is, if you like, claim 2, which is that the full sequence of partial sums converges to S. And we're going to do this by brute force using epsilon M argument.

So remember, to show something-- a limit of something equals S means for all epsilon, there exists a capital M, so that for all M bigger than or equal to capital M, S sub M minus S in absolute value is less than epsilon. So let epsilon be positive.

Since S sub 2k convergence to S, there exists M0 natural number. So that all k bigger than or equal to M0 S sub 2k minus S in absolute value is less than epsilon.

Now, we haven't used at all in this proof that the xn's really converge to 0. We did use that they were non-negative at one point. But now, this is where we'll use a conversion to 0, because here's the intuition. The S sub 2k's are converging to something.

So now, I just have to look at the odd ones. I've shown all the even partial sums converge to S. So if I can show the odd partial sums also converge to S, then essentially, I'm done. What's the difference between an even partial sum and an odd partial sum? Well, it's just x to the 2k plus x sub 2k plus 1, which is converging to 0. So they don't differ by much.

And that's essentially the whole argument right there. So since xn's converge to 0, there exists natural number M sub 1 so that for all n bigger than or equal to M sub 1, x of n is less than epsilon over 2. I should have written Epsilon. Over 2 here.

Choose M to be the maximum of two numbers 2M0 0 plus 1 and M1. So you'll see why I made these choices in just a minute. So suppose m is bigger than or equal to M. We now want to show that S sub m minus S is less than epsilon in absolute value.

So there's two cases. If m is even, then m over 2 is bigger than or equal to capital M over 2, which is the max of these. If I divide by 2, that's certainly bigger than or equal to M0.

And therefore, I use this first inequality, S sub m minus s, which is equal to S2 times m over 2, minus S. Now, m over 2 is an integer bigger than or equal to m sub 0. So I can use this inequality. It's less than epsilon over 2, which is less than epsilon.

And now, we do odd. So again, m is bigger than or equal to capital M. And there's two cases, even and odd. If m is odd, let k be this integer now. Since m is odd, m minus 1 is even, divided by 2, that's an integer. And so, this is-- so m equals 2k plus 1.

And since m is bigger than or equal to M0, this implies a couple of things. This implies that this integer-- so our m, this implies that 2k plus 1 is bigger than or equal to 2. So m is just equal to 2k plus 1. So that's bigger than or equal to m, which is bigger than or equal to 2 m sub 0 plus 1.

And therefore, k, this integer here, is bigger than M0. Also, m is bigger than or equal to m sub 1 as well. I mean, m is bigger than or equal to m. And m is bigger than or equal to the max of these two things. So it's bigger than or equal to 1.

Then if I look at S of n minus S, this is equal to S sub m-- so the nth partial sum is equal to the m minus first partial sum plus the next term minus S. Now m minus 1, in terms of the integer k, is equal to 2 times k. And now, I'm going to take this S and group it with this guy.

And so, since k is bigger than or equal to m sub 0, we can use this inequality here after we do the triangle inequality. So this is less than or equal to S 2k minus S, plus the absolute value of this thing.

So again, since k is bigger than or equal to m sub 0, I can use this inequality to get this is less than epsilon over 2, plus this guy. And since m is bigger than or equal to, capital M, which is bigger than or equal to capital M sub 1, I can use this inequality. And this is less than epsilon.

So we've done the case of m even or m odd, m bigger than or equal to m. So in summary, we've shown that if m is bigger than or equal to m, S sub m minus S is less than epsilon. And therefore, the S sub n's converge to S. And I don't think I have enough time to do this next theorem. So we'll stop there.