[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIGUEZ: So, all right, so let's start talking about some math. So there was a last result I was going to prove for series, which is that if you have an absolutely convergent series, and you rearrange it, then that series is also absolutely convergent and converges to the same thing as the unarranged series. But I'm going to leave that to the lecture notes, and we're just going to start on a new topic.

So the topic that we're now on is continuous functions. And so continuous functions, as we'll see, is a statement about how a function behaves near a point in relation to how it behaves at the point. Now, what it means for a function to behave near a point-- that's the notion of the limit of the function, which is what we'll first discuss.

OK? Now, at what points are we going to be looking at? So this is where limits take place. And somehow, we want to look at a function near points where there's a set nearby to actually look at f at. So f is usually defined on some set S, and we want to look at points so that there's a lot of S nearby. And we kind of already dealt with these types of points when we encountered cluster points in the assignment. So this is exactly where we start looking at limits.

So this is a definition that I'll recall from the assignment. Let S be a subset of R and x and R. We say x is a cluster point of S if, for all delta positive, the interval x minus delta, x plus delta intersect S take away x is non-empty, OK? So the new terminology-- not so new since we dealt with it in the assignment-- is that of a cluster point.

And another way to say this is that-- so an equivalent way of stating this is that, for all delta positive, there exists, let's say, a y in S such that 0 is bigger than x minus y is less than delta, OK? So what this means is that, as far as the picture goes, x is a cluster point if, when I ever take any small interval about it, I can find some S in there other than possibly x. And this should be able to be done for every interval.

So let's take a look at some examples. Let's say we take S to be 1/n in a natural number, OK? And so since now I actually have people in front of me, I can actually ask questions.

So whoever first says it, what would be a cluster point of this set S? So what would be a point in the real numbers So that there's a lot of S near it? Feel free to blurt it out if it comes to you.

Why don't you just take a guess? 1? So that's one possibility. So there has to be a lot of S near this proposed cluster point. So let me mark out the points that we have. There's a-- I'll put 0, and then 1/2 is there, and then 1/3, 1/4, 1/5, 1/6, so on, and so on.

And so I see a guess of 0. So that's a good guess as well. So why would 1 maybe not be a cluster point? Well, it's because if I draw a little interval around 1, then there's only one point in S in here, and it's the proposed cluster point, 1, OK? So remember, what we should be able to do is find in every small interval [? sum ?] of S in the interval that's not equal to the point, OK? And we can't do that for the point 1.

But for 0, if we draw any interval around 0, there's plenty of S in the interval not equal to 0, OK? So here, 0 is a cluster point of S. And let's give a quick proof of this. So we have to verify for every delta positive this interval, 0 minus delta, 0 plus delta intersect S take away 0 is non-empty.

So the delta would be positive, and now here's the picture. Here's 0. Here's delta, minus delta. We have to show that there's some S in here. So we simply choose natural number n large enough so that 1/n is there, and we can always do this because of the Archimedean property of the real numbers, right?

So let delta be positive. Choose natural number n so that 1/n is less than delta, and it's certainly bigger than 0. So then n is a natural number. Then 1/n is in x minus delta, x plus delta intersect S take away 0. So this should have been 0, which implies this set is non-empty because it has something in it, OK?

All right, was that example OK? Did you all get that all right? OK, so feel free to ask questions as they come up. So let's do another example. This one, I will not give a full proof of, but we'll just kind of talk our way through it.

Let's say S is now the open interval 0, 1. What is the set of cluster points of 0, 1 equal to? So what would be a reasonable guess or any guess?

Yeah, so, certainly, this will contain 0, 1, right, since-- so 0, 1, everything between 0 and 1 should be a cluster point of 0 and 1 because there's always a lot of numbers. I mean, there's a lot of numbers between 0 and 1. So if I draw this 0, 1, and [? have ?] everything in between 0 and 1 is certainly a cluster point. What about the point 1? Is that a cluster point of this set?

That's a good observation that, yeah, it's a sup of the set. So for every-- right, so for every delta positive, if I draw a little interval around 1, there's going to be a number less than 1 that's bigger than 1 minus delta, right? The set of cluster points of the open interval 0, 1 is the closed interval 0, 1. OK?

Now, let's do another example. Let's say we look at now, instead of rational numbers-- so you should, again, think of cluster points as being the set of points where there is a lot of this set nearby, OK? Now, what is the set of cluster points of the rational numbers?

OK, so perhaps it will contain the irrational numbers. So let's think about this for a little bit. So, for example, let's look at square root of 2, OK? Perfectly good rational number-- I mean, irrational number. If I draw a little interval around square root of 2, can I find a rational number in there that's not equal to a square root of 2?

Right, because we have this theorem about the density of the rational numbers, right, that we have-- so remember that this was a theorem we proved at one point. For all x, y real numbers with x less than y, there exists a rational number r such that x is less than r is less than y, right? So we can always find a rational number here, or we could have taken another rational number here, yeah?

So for every interval we draw around square root of 2, we can find a rational in there not equal to square root of 2, obviously. But we can find a rational in that interval, yeah? So this suggests that the set of cluster points contains the set of irrationals.

What about rational numbers? Are those also cluster points? Let's say I take 0. Now I look at a small interval around 0. Can I find a rational number in there other than 0? I'm not asking like, actually give me one. I'm just saying, does there exist a rational number in this interval other than 0?

AUDIENCE: Yeah.

CASEY RODRIGUEZ: By this theorem, right? You can also use the theorem to find irrational in this interval as well. So no matter if it's square root of 2 or any irrational number, no matter if it's a rational number, everything is a cluster point. So the set of cluster points of rational numbers is equal to R, OK? Is that clear? OK, feel free to stop me if you have any questions.

So now let's look at, for example, S equal to just a single guy, say, 0, OK? Remember, cluster points of a set means there's a lot of that set nearby. Now, I claim that this guy has no cluster points because if you think so intuitively why is this, this is because, for something to be a cluster point, it has to have a lot of the set near it, a lot. And there's not a lot of set to begin with. There's just one element.

So let's see why this is the case, OK? And this gives us a chance to negate the definition of cluster point, which is always a good idea. OK, so S equals just the singleton set, 0. This has no cluster points, and so the negation of the definition of a cluster point-- so x is not a cluster point of S means if this is some bad delta-- remember, so the definition of cluster point is for all delta.

So negation means there exists one bad delta so that x minus delta 0, x plus delta 0 intersect S take away x equals the empty set, OK? So the picture that goes along with this is that x does not cluster point if there's some interval and possibly the only S there is the point x, but nothing else, OK?

All right, so let's use the negation of this definition to show that there are no cluster points, OK? And we have to deal with two cases. I want to show x is in R, and x is not a cluster point of S. And I'll just do the case x not equal to 0. x equal to 0 is much easier, so we'll just do x not equal to 0.

Let's prove this. And what's the idea? We have to be able to find-- so here's 0. This is all that S is, is just this point, 0. Now we have a point x. What would be an interval containing x that contains none of S?

So x here is just some nonzero number. S is what's highlighted. The interval around x that doesn't contain any S. So like a x/2? 3x over 2, meaning delta equals x/2, at least for this picture, yeah? OK, and that's essentially the proof.

So choose delta 0, and we'll simplify this further and suppose x is positive. For x less than 0, you choose delta 0 to be the absolute value of x. But for this one, this is simple enough-- choose delta 0 to be x/2 and x minus delta 0, x plus delta 0 is equal to x/2, 3x over 2.

And since 0 is not in this set, in this set, [? implies ?] intersect S take away x equals empty, OK? So we had to find a delta so that this interval around x doesn't contain anything from S, and we've done that by choosing delta 0 to be x/2.

OK. So I could have done it for just-- so I did it for a set containing just one element, showed that this set has no cluster points. You can also show that if it has finitely many points, it has no cluster points. And so therefore, any finite set has no cluster points, but being a cluster point has nothing to do with the cardinality.

So if you see the example in the notes, which you can think about, and I'm not going to go over right here, but, for example, x equals z, the set of integers-- this has no cluster points, OK? So being a cluster point means, where is this set clustering? So it has nothing necessarily to do with the size of the set but where the set is taking up space in the real number line.

All right, so with the definition of cluster points, let me first state the theorem that, in fact, you proved in the assignment, which is that for cluster points-- let S be a subset of R, and x is a cluster point of S if and only if there exist sequence xn of the elements of and S take away x such that xn converges to x, OK? And that was in the assignment. You proved that basically using the squeeze theorem in the definition.

All right, so with the notion of a cluster point, these are the points in a set where we're going to start talking about how a function behaves near a point, which is the notion of a limit of a function is our next definition. Let S be a subset of R. And even though I was referring to cluster points earlier as x, we're now going to switch over to c.

Let c be cluster point of S. And let f be a function from S to R, all right? So you say f of x converges to L as x goes to c. So this is the new terminology here.

So here, L is a real number if, for every epsilon positive, there exists delta positive such that if x is in S, the absolute value of x minus c is bigger than 0 less than delta, then f of x minus L is less than epsilon.

So let's try and compare this definition a little bit to what happens for a sequence. So for a sequence, the sequence is getting close to L as long as we go far enough out in the sequence, as long as we look at terms far enough out. Now, for limits of functions, that going far enough out along the sequence is replaced by, as long as we look close enough to the point c but not at the point c.

So if you'd like, this says that if x is near c-- and so this is just an intuitive interpretation of what this definition says. If x is near c, then f of x is near L. And let me also throw in here that we're saying that x is near c but not equal to c.

But for a limit, you don't look at what happens at f of c. The function doesn't even need to be defined there to be able to define what a limit is. We just care about what happens nearby, OK?

So you could have-- the picture you should have in mind is that here's c, and maybe f is not even defined up to there, but there is f, the graph of the function f there. There's L, and as long as I get very close to c, I'll be very close to L, "very close" being measured within some strip here, OK?

OK, so this is the notion of the limit of a function as we approach a cluster point of the set S where the function f is defined. Just a little bit of notation-- typically write like we did with sequences f of x converges to L as f of x arrow L. And then I'll write, as x goes to c. Maybe I won't write as x goes to c. We note this by limit of f of x equals L as well.

OK, so first question about limits is, are they unique? And this is, in fact, why we require that c be a cluster point of S because if we didn't require c to be a cluster point of S in this definition, limits would not be unique. In fact, you could have functions converging to whatever you want because the definition would then be vacuous.

But if you don't want to think about that, that's fine. It's not a big issue. But I'm just going to point out at one point, we're going to use the fact that c is a cluster point to prove the following theorem.

So let c be a cluster point of S, which is a subset of R. And let f be a function from S to R. If f of x converges to L1 as x goes to c, and f of x converges to L2 as x goes to c, then L1 equals L2, OK? So this is what I mean by uniqueness, that a function can only have one limit as it approaches a point c.

Let me give the proof. So we're going to play this game again, where instead of showing something as 0 or something is equal to something else directly, we'll give ourselves a little room and prove the following. We'll instead show that while epsilon positive, L1 minus L2 is less than epsilon. So L1 and L2, these are just two fixed real numbers. If their absolute value is less than epsilon for arbitrary epsilon, then this is a number which is smaller than any positive real number. That implies that that number is 0, so L1 equals L2, OK?

Why we give ourselves a little room is because, in the definition, we have a small parameter there which we hope to use. So let's prove this. Let epsilon be positive. So since f of x converges to L1, there exists-- so just from the definition, given some small tolerance, I can find a little number delta so that if I look inside this interval, I get mapped to the interval L plus epsilon, L minus epsilon.

So there exists a delta 1 positive such that if x is in S-- and I'm going to probably keep forgetting to write that, but that should be understood that x is in the domain of the function f. If x minus c is less than delta 1 and bigger than 0, then f of x minus L1 is less than epsilon/2, OK?

And the same thing for L2. I mean, if f of x is converging to L2, then there exists a delta 2 so that I have the same statement. So instead, I'm just going to change this 1 to i, i, i, i. And here, i equals 1 too. So since f of x converges to each of these two numbers, L1 and L2, there exists a delta 1 and a delta 2 so that if I'm within this tolerance delta of 1 or delta 2, then f of x minus delta 1-- L1 or L2 is less than epsilon/2.

Let delta be the minimum of these two numbers. Now, since c is a cluster point of S, in fact, there does exist an x in S such that 0 is bigger than x minus c is less than delta, all right? And I'm going to use this x to compare L1 and L2.

Then since x is in this interval, or I should say, since it's-- so remember I had this, which means I have this inequality or i equals 1, 2. I get f of x, or I should say, L1 minus L2. This is equal to if I add and subtract f of x and now use the triangle inequality and use the inequality that's in green, the f of x minus L1 is less than epsilon/2.

And same for L2, I get-- equals epsilon, OK? And that's the end because remember we started off with the absolute value of L1 minus L2. Are there any questions about that?

OK, so not the most exciting theorem in the world that this thing that we've defined. A limit is a unique thing for function. But let's start looking at some examples. And at one point, we'll look at the negation of this definition.

So let's suppose-- so let's look at the limit as x goes to c of the function ax plus b. And I claim this is equal to ac plus b, OK? So here, S is, although I didn't write it out, this function ax plus b, this is-- I'm looking at it defined on all of R, and c is a real number, OK? So we have to verify this definition.

And since it's for every epsilon, there exists something. The beginning of every proof, I'll give you a few points if you can just say, let epsilon be positive. How will we choose delta? Well, you do similar computations as you would if you were doing epsilon m arguments for sequences.

So what are after? We want to find delta so that x minus c less than delta and also bigger than 0 implies f of x, which is ax plus b, minus ac plus b is less than epsilon. So let's start off with this thing, play with it, and see if we can find how to choose delta. Remember, we don't solve this inequality. We want to find how to choose delta based on estimating this thing and trying to make it less than epsilon.

So f of x minus ac plus b, absolute value-- this is equal to if you just plug in f of x equals ax plus b, this equals a times x minus c, which equals absolute value of a times x minus c. And now we're in business because the thing that we do have control over is the delta, right? It is how big x minus c is. So this is less than absolute value of a times delta if x minus c is less than delta.

And remember, the thing that we want in the end is the thing in yellow, and the thing we get to control is the thing in green. So if I want the thing in yellow, and I have the thing in green, how do I choose delta?

So in the end, I want f of x minus ac plus b to be less than epsilon. Right now, I'm at the absolute value of a times delta. Delta is the thing I get to play with. So how do I choose it? Epsilon to be absolute value-- right? Because if I choose delta this way, then x minus c less than delta implies that, just by this computation here, I would get epsilon in the end, yeah?

So that's kind of the scratch work that goes into it. Again, delta's the thing you get to choose. You get to choose based on epsilon. And so you look f of x minus ac plus b. You estimate it to be no bigger than the absolute value of a times x minus c, which is less than the absolute value of a times delta. And so if you want that thing to be less than epsilon, you should choose, for example, delta to be epsilon over the absolute value of a. There's not a unique choice. I could have chosen delta to be epsilon/2 absolute value of a, but this is a choice, OK?

So let epsilon be positive. Choose delta to be epsilon over absolute value of a. If x minus c is less than delta, then-- and when you write the proof, it's essentially a rehashing of the best parts of your computation. And f of x minus a times c plus b, this is equal to absolute value of a times x minus c.

So if we take f of x and subtract our proposed limit ac plus b, we get the absolute value of a times absolute value of x minus c, which is less than the absolute value of a times delta, which equals epsilon since we chose delta to be epsilon/a, OK? That's the proof. Are there any questions?

OK, so let's do another example. Let's look at the limit as x goes to c, a square of the function square root of x. And I'm going to show this is equal to square root of c. And here, although I'm not writing it, S will be, for us, the domain of the square root, which is the closed interval 0, infinity, not including infinity, of course. And for what I'm going to do, c is positive, OK? You can also do c equal to 0, but for this proof, I'm going to do c positive.

All right, so I have to verify this definition, which means for every epsilon, I can find a delta, so let epsilon be positive. Let's go over to our box of scribbling and scratching. So remember, our goal is to find a delta which I can choose so that f of x minus square root of c, in this case, is less than epsilon. So if I look at f of x minus square root of c, this is square root of x minus square root of c.

And perhaps you remember from your days in calculus that whenever you see the difference of two square roots, it's a good idea maybe to multiply the top and bottom, or, I guess, multiply by 1 in a very special way, so that I get the difference of squares. So this is equal to the square root of x minus square root of c times square root of x plus square root of c all over square root of x plus square root of c, OK?

And so now I have-- on the top, I have product a minus b times a plus b. So that's the difference of squares, which gives me x minus c on top over square root of x plus square root of c, OK? And remember, this is less than the thing that we're going to choose in the end. The absolute value of x minus c is less than delta, so this is less than square root of x plus square root of c.

Now, in the end, you have to only-- you can choose delta depending only on epsilon and maybe the point c, OK? You cannot choose it to be dependent on the point x, which is changing, right?

What maybe you're tempted to do here is to then choose delta to be square root of x plus square root of c times epsilon. Don't do that because delta, again, only depends on the point c and the point epsilon. It cannot depend on the thing that's changing, which is x, right? This is changing. So you don't make this choice.

Now, if we just do one more thing, then we'll get to something where we can choose an independent of the thing that's changing, x. The square root of x on the bottom is only making things smaller because that's non-negative. So this is less than or equal to delta over square root of c, OK?

So now, in the end, I have something which is delta and depends on the point c, and I want f of x minus square root of c to be less than epsilon. So how should I choose delta? Shout it if you know it.

Right, epsilon times the square root of c. So now this will be our choice. So choose delta to be epsilon times square root of c.

Now we have to show this delta works, which is essentially, again, just rehashing our computation over on the right, which gave us how to choose delta than if the absolute value of x minus c is less than delta, f of x minus square root of c-- and I'm not going to write f of x.

I'm going to write the square root of x is equal to basically what we did over here in the box, which is absolute value of x minus c over square root of x plus square root of c, which is less than or equal to, if I take away the square root of x on the bottom like we did before, which is less than delta or square root of c, which equals epsilon square root of c over square root of c, which equals epsilon.

So this delta works. OK, Is there any questions about that example? OK.

So let's do one more example, which really illustrates what I was getting at when I was talking about the limit only cares about what the function is doing near a point but not at the point. So let's say we look at the function f of x given by 1, if x equals 0, and 2, if x does not equals 0. So this is-- here's the point 0. And where it's not equal to 0, it's equal to 2, and when x equals 0, it's equal to 1, all right?

And they claim that the limit as x goes to 0 of f of x-- this equals 2. And the thing to note is that this limit it does not equal f of 0, which it did for the previous two cases, right? If we look back at these two examples that we did a minute ago, the limit as x goes to c of both of these examples was I just plug in c to the function, right?

But for this example, that's not the case. The limit is not equal to the function evaluated at the point in highlighting that limits don't care about the function evaluated at the point. They care about what's happening near the point, OK? And near the point, f of x is just equal to 2 identically.

So I'll give you the quick proof, or you can completely ignore it and just believe that the limit as x goes to 0 of this function equals 2. So the point is that for x not equal to 0, f of x is just a constant 2. So let epsilon be positive. And you can choose delta to be whatever you like because f of x when x is not equal to 0 is going to be equal to 2, the proposed limit.

Then if 0 is bigger than the absolute value of x minus 0, which is just x, is less than delta, then this implies x is not equal to 0. And therefore, if I look at f of x minus 2, this is just 2 minus 2 equals 0, which is less than epsilon, OK?

Any questions about this example? No questions?

All right, so what we'll talk about now is-- so before this section on continuous functions, or at least this section on limits of functions, we had the notion of limits of sequences. And so a natural question is, how do limits of functions and limits of sequences belong together, interact, relate to each other, I guess, is the best way to phrase that question.

And the point is that to decide if a function has a limit it, suffices to stick in sequences that converge to that point that you're looking at. And that's the content of the next theorem. So let S be a subset of R, c cluster point of S. Let f be a function from S to R.

Then the limit exists. Limit as x goes to c of f of x equals a number L. This is equivalent to the statement that for every sequence x sub n of elements of S take away c such that xn converges to c, we have f of xn converges to L, OK?

So again, the statement of this theorem is that a function converges to L if and only if for every sequence I stick into the function which converges to c, f of x then converges to L. So to choose or to decide if a function converges, it suffices to look at convergence of certain sequences, OK? OK.

And why is this theorem important, or at least very useful? It's because with this theorem, we now get basically every analog of theorems we proved for sequences for free, for example, limits of sums of functions, a squeeze theorem that can be stated for limits of functions, and so on. And I'll say a few comments about this after we proved the theorem.

All right, so let's-- this is a two-way street. And I'm going to mark the second statement in green. That way, I don't have to rewrite it again. So first, assume that limit at x goes to c of f of x equals L, and now we want to prove the statement in green, OK?

So here's the idea. So now we want to show-- so let xn be a sequence of elements in S take away c such that x sub n converges to c. And what we want to show is that f of xn, which is now a new sequence converges to L as n goes to infinity. OK, so here, you should write, as n goes to infinity, in here, OK?

So let me move that, and let me draw the picture of why you should believe this. So remember, what it means for a sequence to converge to a number means, for every epsilon, there exists a capital M so that for all n bigger than or equal to capital M, the sequence is close to the number L within epsilon distance to it.

So let's think about this a little bit. I'm not going to draw the graph. I'm just going to draw two copies of the number line. Here, we have the number L. Here's the number c and L plus epsilon, L minus epsilon.

So let epsilon be positive. And the idea is we want to be able to find or say there exists a natural number capital M so that n bigger than or equal to capital M implies f of xn is inside this interval here.

Now, since f of x converges to L as x converges to c, there exists a delta so that-- so this is the picture that goes along with it-- so that if I'm inside this interval here, then I get mapped into this interval here. So this is what I get from the assumption.

So then how I should choose the natural number capital M is so that all of the x sub n's lie in this interval, c minus delta, c plus delta. And if I can arrange that, which I can because the x sub n's converge to c, then I will get that these guys get mapped into this interval, L plus epsilon, L minus epsilon, OK? And that's the whole idea of the proof. Now we just need to write it down.

Since f of x converges to L, exists a delta positive such that if less than delta, then if x minus L is less than epsilon. That's what I drew in yellow there. Now, since the sequence x sub n equals c, this implies that there exists some natural number M sub 0 such that little n bigger than or equal to M sub 0-- since all of these x sub n's are not equal to c, by assumption, they're in the set S take away c.

X sub n minus c, and absolute value is bigger than 0 and less than delta, OK? This follows from the definition of convergence of sequences to a point. Here, if you like, I'm choosing epsilon in that definition to be delta. And this delta is coming from the definition for convergence of functions.

And so we'll choose capital M to be this integer M sub 0. And if n is bigger than or equal to M, I get that, by our choice of it being M sub 0, you get that x sub n minus c is less than delta. That means that it falls in this little yellow highlighted part in the first number line, which implies-- so here, what I'm using is what's highlighted in blue.

So this is content of this implication-- implies that f of xn minus L is less than epsilon, all right? And that's what we wanted to do, right? We wanted to show that there exists some natural number M so that n bigger than or equal to M implies f of x minus L is less than epsilon in absolute value. And therefore, we've proved that the limit as n goes to infinity of the sequence f of xn equals L.

OK, so that's one direction of this if-and-only-if statement. Let's prove the opposite direction. So suppose what's highlighted in green holds. OK?

And this proof is going to be by contradiction. So let's suppose that our assumption, what's in green, holds. But the limit as x goes to c of f of x does not equal L. And now we're going to do the proof by contradiction, meaning suppose the outcome is false. OK?

We're still assuming what's in green holds. That's our assumption. And for proof by contradiction, we assume that the outcome or the conclusion is false, which is that the limit as x goes to c of f of x does not equal L.

Now, what does this mean? So now we get to negate the definition and get better acquainted with it. This means that-- so the definition of limit, if you go back to the notes is, for all epsilon, there exists a delta statement. So the negation is that there exists some bad epsilon, epsilon 0, such that for all delta positive, there exists an x such that x is close to c, but f of x is far away from L. OK?

So the negation of the definition of limit of something being the limit is that there exists a bad epsilon so that no matter how close I get to c, I can find something very close to c that's far away from L if I stick it into f, OK? That's the intuitive way of viewing the negation of this definition.

OK, so now I'm going to apply this with delta chosen to be 1/n for in a natural number, OK? And this says-- so this statement here is a for-all-delta statement. So that holds for every delta I pick.

So for all n, a natural number, there exists an x sub n such that 0 is bigger than x sub n minus c is less than delta-- I'm going to pick to be 1/n-- and f of x sub n minus L is bigger than or equal to epsilon 0.

So first off, I made a for-all-n statement. Let's think about this for just a second. So certainly, for delta equals 1, I can find a number x sub 1 satisfying 0 is less than x minus c is less than 1, and f of x minus f of x sub 1 minus L is bigger than or equal to epsilon 0. But delta can be anything I choose, so for delta equal 1/2, I can find an x sub 2 such that x sub 2 minus c is less than 1/2, and f of x minus L is bigger than or equal to epsilon 0.

So I'm just choosing delta to be 1/n for each n, a natural number. Choosing delta to be 1/n, OK? Is this point clear? OK, please stop me if you have questions and something is not clear.

OK, so I have this sequence of elements of S that are not equal to c because they are an absolute value bigger than 0, or the absolute value of x sub n minus c is bigger than 0. And they satisfy these two inequalities, OK? Then let me just write this inequality again.

So then the inequality that I have is, for all n, the natural number 0 is less than x sub n minus c is less than 1/n. I claim that the x sub n's must then converge to c. What tool can I use to say that?

Squeeze theorem, OK? Because this thing on the left, 0-- this, if you like, is a sequence, just a constant sequence. This converges to 0. And 1/n, something on the right, converges to 0 as n goes to infinity. So that implies the thing that's sandwiched in between gets squeezed to 0. So that implies limit as n goes to infinity of x sub n minus c equals 0, i.e., xn's converge to c.

Now, I'm assuming what's in green holds, which is, if I take any sequence converging to c, f of xn has to converge to L, OK? So by what's in green has given me this implication-- 0 must be equal to limit as n goes to infinity of f of x sub n minus L.

But What. I know is that, since I've assumed f of x does not converge to L, along this sequence, each of these guys is bigger than or equal to epsilon 0, which is positive. So now I've concluded that 0 is bigger than 0. That's a contradiction, OK?

OK, so next time, we'll talk about the applications of this theorem. Do you all have any questions about this proof or anything that we've covered so far today?

You mean how you get the choice of delta from doing--

AUDIENCE: Yeah.

CASEY RODRIGUEZ: Right. So let me phrase this a-- Yeah, I mean, so-- and probably in most classes, you would just be given this proof that's over here on the left. And somehow it looks like magic that I chose this delta this way, and it ends up working, right?

But the thinking that goes in behind it is-- so what's the thing you want? You want what's in yellow. So let me erase some of what's in yellow. You want to choose delta so that if you take f of x minus ac plus b, that thing in absolute value will be less than epsilon, OK? So start with that thing that you want to make small and that you want to ensure is small, f of x minus ac plus b, and start computing.

And when you compute it, you get what we got here in the second bit of yellow. And we're assuming that the absolute value of x minus c is less than delta, right? We're trying to choose delta so that what's in green implies what's in yellow. So far, we haven't chosen delta. We just know that the absolute value of x minus c will be less than delta. And we want to choose delta so that that implies what's in yellow after the arrow there.

So when we do this computation, and assuming that the absolute value of x minus c is less than delta, we get this thing right here, the absolute value of a time delta. This is just assuming that the absolute value of x minus c is less than delta, which is what will it be assuming.

And we want what's in yellow, which is that thing to be less than epsilon. So what we've arrived at is this thing here, which we want less than epsilon, is less than this thing in red, OK? And therefore, we want to choose the delta so that the thing in red is less than epsilon or less than or equal to epsilon. That will ensure that f of x minus ac plus b in absolute value will be less than epsilon, OK?

So I'm not solving an inequality. I'm not solving the inequality I want, which is f of x minus ac plus b. I'm starting with what's on the left, estimating it using my assumptions, and then, at the end, choosing delta so that I come up with epsilon in the end.

So in the last step, I have absolute value of a times delta. And I want that to be less than epsilon. So I could have chosen delta to be less than or equal to epsilon/a. So anything, any delta like that would have worked. So I could have put delta to be epsilon over 2a. That still would have worked, or 3a, OK?

Yes, [? what ?] [? they're ?] green. That's the thing we're doing, thing we get to use in this computation. Are there any other questions?

This is a style thing, all right? So throughout this entire semester, for all epsilon, you have to be able to find the M so that something happens, right? So at some point, you always say, choose M to be something, right? Maybe it's the maximum of some M0, M1 that you had. Or choose M to be so that 1/M is less than epsilon or something like that.

So I'm just sticking with that style of saying, choose M to be this. Otherwise, I could have said, there exists a capital number M so that for all n bigger than or equal to capital M, x sub n minus c is less than delta, and then move to the next part. But throughout the course, I've always, at least in the proofs I presented, you're making a choice of capital M, right? It's for all epsilon, there exists a capital M. So that means you have to tell me how to choose capital M.

And it's the same thing with these epsilon delta proofs is that, for all epsilon, there exists a delta. So at some point, you have to tell me how to choose delta in the proof. So I've just kind of stuck with that style of saying, choose M to be something, although you're absolutely right that I could have just said, for all n bigger than or equal to capital M0, I have what I want. And therefore, implicitly, I'm saying that M equals M0 is the thing that works.

Yeah, that's the same delta. Yeah, that's the same delta. Mhm.