[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIGUEZ: So we ended last time with these elementary properties that you know about the absolute value. So let me prove one more theorem about the absolute value, which maybe you didn't cover in calculus. But it's of fundamental importance in real analysis, which is the triangle inequality, which states that for all xy in R, the absolute value of x plus y is less than or equal to the absolute value of x plus the absolute value of y.

So why is it called the triangle inequality? Well, let me-- instead of x and y being real numbers, let's think of these as vectors. So there's x. Let's say y. And then vector x plus y is this side. Then what this is stating is that the length of one of these sides of this triangle is less than or equal to the sum of the two sides, the sum of the lengths of the other two sides of the triangle. So that's why it's called the triangle inequality.

OK, so the proof is not very difficult, although this inequality is probably one of the three most useful things you'll learn in this class if you go on to study more analysis-- the other one being-- OK, so I'm coming-- this is knowledge from PDE. So I guess I'm viewing it through that lens of what I find useful in PDEs. But triangle inequality, integration by parts, change of variables, those three power the analysis machine that I'm most familiar with.

So how to prove the triangle inequality-- if x and y is an R, then clearly, by property number 6, the x is less than or equal to the absolute value of x and y is less than or equal to the absolute value of y. So their sum is less than or equal to the absolute value of x plus y.

And not only that, I can replace x by minus x and I get minus x plus minus y, this is less than or equal to the absolute value of minus x plus the absolute value of minus y. But this is just absolute value of x plus y.

And both of these therefore imply-- so if I just take this and multiply through by minus 1, this inequality, which, by number 5 in the properties that we proved, for the absolute value implies that the absolute value of x plus y is less than or equal to the absolute value of x plus the absolute value of y.

So let me just make a remark about the reverse triangle Inequality, which will make an appearance in your assignment. And so this involves the absolute value of x minus y and the absolute value of the difference in the absolute values. Why is it reverse? Because the inequality reverses.

The absolute value of x minus the absolute value and y, take the absolute value. This is less than or equal to the absolute value of x minus y. And you'll prove this in the assignment. So these two-- so this follows from the triangle inequality. But these two inequalities get used quite a bit. And we'll use them quite a bit throughout the course.

All right, so let me take a minute to reconnect R with what you know and love from calculus, namely decimals. And we're going to use this fact about R to answer a natural question that maybe you're asking yourself, which is the following. So let me raise this as a question.

So this first part is not the question. We've already addressed this in the first assignment, the fact that the rationals is countable. In fact, you did it for the positive rationals. But if the positive rationals are countable, you can then prove that all rationals are countable. And a natural question-- is R countable? Or is the set of real numbers countable?

So in the end-- I'm not going to leave you in suspense-- the answer is no. And we're going to prove this. But let me just make a few remarks about what this says. So it's the following. So if Q-- so this is not a proof of this. I'm just going to make a remark based on this answer.

So since R is uncountable and Q is countable, this implies that R takeaway Q, the set of irrational numbers, is, in fact, uncountable. Why? So I'm just going to say in words why this is true.

So suppose otherwise that both the rational numbers and the irrationals are countable. Then you can map in a bijective fashion the rationals to the natural numbers and the irrationals also to the natural numbers. But then you can map those natural numbers to 0, minus 1, minus 2, minus 3, and so on. So then you find that there is a bijection from Q to the natural numbers and the irrationals to the integers less than or equal to 0.

And therefore, there's bijection from R to the integers. But the integers are countable, which would contradict the fact that R is uncountable. Therefore, the set of irrationals is uncountable. So in short, if you didn't follow that, the set of irrationals is uncountable because otherwise this would force R to be countable.

Now, I haven't proved yet that R is uncountable. We'll do that in just a minute. And we'll use a theorem about decimal expansions for-- or decimal representations of real numbers.

So now, connecting rationals and irrationals to what you've seen since you were small, so to decimals, we simply-- we typically-- we think of rational numbers as in terms of decimal representations. What does that mean? That means we take-- if x is in Q, then we'll write x as some number times some digit times 10 to the k plus 10 to the k minus 1 plus d0.

And then we go to the tenths spot, so d to the-- d minus 1 10 to the minus 1 plus-- so this is discussion. I'm not stating any theorems right now, until we get down to maybe some last decimal spot. And with these digits, a subset of 0, 1, 9.

So all I'm saying is typically we think of x as this way. And then we write x as dk, dk minus 1, d minus 1, d minus 2, and so on. I'm using this notation because I'm going to be-- this is the notation I need to state the theorem about the real numbers. But it's quite silly that I'm about to write this down in this setting. Welcome to MIT.

So for example, 1 times 10 plus 1 times 10 to the 0 plus 1 times 10 to the minus 1. This is supposed to represent the number 11.1, which is the rational number.

It's quite funny that I had to think about that for a minute to go from the decimal expansion to the rational numbers. But don't give me too hard of a time. In a lot of at least pure math, we typically write things in rational numbers. We don't write in decimal-- I haven't had to write in decimal expansion in quite some time.

Anyways, now, of course, not all rational numbers can be written as a finite decimal expansion. For example, 1/3 you have to write as 0.33333 and so on and so on. So that's not necessarily covered under the discussion that I have right here. But it will be covered in what I'm about to say for the real numbers.

So we could cover 1/3. And then we can also cover every real number if we allow infinite decimal representations. So I'll use the word decimal expansions or decimal representations interchangeably there. So let me make a definition. And what I'm going to state is about real numbers between 0 and 1. Of course, if you want to talk about real numbers bigger than 1, then you just tack on 1 to the-- if you like, to the d0 place.

So the infinite part is having to deal with what comes after the decimal point. So the let me make the following. So this is a rigorous definition. Let x in 0, 1 excluding 0. And let d minus j be in 0, 1 to 9. for j, a natural number.

So we say x is-- so what does it mean to say that x is given by an infinite decimal expansion or given by an infinite decimal representation? We say x is represented by the digits.

And we write x is equal to d minus 1, d minus 2, d minus 3. If-- so this is the precise meaning of this. If x is equal to the supremum of the set-- so I take a finite decimal expansion.

And then so I get a number for each n, a natural number. So the first one is just d minus 1 10 to the minus 1. And then for n equals 2, I tack on next hundredths place, and then the thousandths place. And so I get a set of real numbers. If I take its supremum and I get x, then I say that x is represented by these digits or that x-- or that this gives a decimal representation of x.

So that you see this lines up with what you know, if I look at 0.250000, remember this is now the sup of what? Of a certain set. So let me do first this guy. So this is 2/10. And then now, I add the hundredths place plus 5/100, and then 2/10 plus 5/100-- this is a set, remember-- plus 0/1000 over plus-- and at each next guy, I'm supposed to take this and add on another 0.

So as a set, this is just a set containing two elements, namely 2/10 and 25/100. And so this is bigger than that guy. So the supremum is 25/100, which equals 1/4. OK, so all of that to tell you that, yes, this definition should-- or at least does ring true with what you were taught, that 1/4 is equal to 0.25, all right?

OK, so now that what we're done with the triangle inequality, I'm going to erase this. And so this is the last bit of discussion about the elementary properties of the real numbers. Next, we'll be talking about sequences of real numbers.

OK, so all right, so I have this definition. The fundamental theorem about the real numbers is that every real number, at least in this setting between 0 and 1, can be represented by a set of digits or it has a decimal representation. And every decimal-- if you like, every set of digits corresponds to a decimal representation of a real number. So let me state the second thing I said first.

So for every set of digits d minus j with each of these d minus j's in 0 up to 9, then there exists a unique x in closed interval 0, 1 such that x is equal to d minus 1-- with this decimal representation.

So you give me some digits and I can find a unique real number that has that decimal representation. Now, before I state the second part of the theorem, this says that I give you digits, I get a unique real number. Now, you ask about the other direction. If I have a real number, do I get a unique set of digits?

And of course, that is not true necessarily because, for example, 1/2, this is equal to 0.500 and so on. But this is also equal to 0.4999 and so on. So yes, every set of digits gives me a unique real number. So if I gave you these digits, they would spit out 1/2 and only 1/2. It wouldn't spit out 1/4.

But if you give me a real number, there does not necessarily exist a unique set of digits giving a decimal expansion for that number. So I can have two different decimal expansions. I have 0.5 and I have 0.4999. I will always have at least one, but it's not necessarily unique.

But we can single out one unique choice by requiring that, in some sense, if I were to truncate it, then the decimal expansion would be less than the number I'm looking at and wanting to expand. So that's the second part of this theorem, is that for every x in 0, 1, now excluding 0, there exists unique digits-- and as before, these are in 0 up to 9-- such that x is-- such that these give a decimal representation of x.

And so what singles out a choice if we're ever coming up against these two things? And if I truncate the decimal expansion-- so I'm not going to write zeros after that. I'm just going to truncate it-- this always is less than x and this is less than or equal to plus 10 to the minus n.

All right, so this inequality will always hold. And in fact, so this inequality will always hold. And in fact, x is always bigger than or equal to this side, no matter what. If x is given by a decimal representation, so we always have these two. This is a less than or equal to and this is a less than or equal to. But we single out a unique choice of decimal representation by choosing this to be a strict less than. So then that forces us to choose this guy over this guy.

So for theorem 2, this would say that the unique representation that satisfies this for 1/2 is given by 0.4999. And for example, for 1 would be just 0.9999 because if I take this representation and cut it off, this is actually equal to 1/2 and not less than 1/2.

OK, so that was a lot of discussion about the theorem. But that was for a reason because I'm not actually going to prove this theorem. It's not entirely difficult. It just uses the least upper bound property of the real numbers. But it's a little clunky to write down. And what I'd rather do is just use this theorem to prove my answer, the answer to my question up there, on whether R is accountable or not. In the textbook, there is a proof of this theorem, which you can read.

OK, so we're actually going to use this theorem to prove the following theorem due to Cantor. So we just went from absolute values, which maybe it was not too shock and awe, now to this theorem, which to me is shock and awe, is the following, that the interval 0, 1 is uncountable.

All right, so maybe you're asking yourself, well, that's showing this is uncountable. And that's saying R is uncountable. So where do we go from this to that? Well, I'll let you think about that. But let's assume this was uncountable and R was countable. So I'm just going to say this, say why out loud.

If this was uncountable and R was countable, then since this is a subset of R, that means this under would be-- so let me just write. Why does this imply-- And I'm going to give you a fake proof, but you can actually make this completely rigorous-- fake proof in the sense that I'm just going to write symbols that imply this.

So why is R uncountable then if this is uncountable? Well, the function that takes an element in here just into R, this is clearly bijective. so just the identity map from 0, 1 into R is an injective map. So I clearly have that the cardinality of this set is less than or equal to the cardinality of R.

In fact, maybe you can think about it. Maybe I'll put it on the assignment. But in fact, one can prove they have the same cardinality. But I have that. And this is uncountable. Therefore, its cardinality is bigger than the natural numbers. So this should imply that cardinality of the natural numbers is less than the cardinality of R. In other words, R is uncountable.

All right, so let's prove this theorem due to Cantor. Now, this other theorem due to Cantor, that the cardinality of the power set is bigger than the cardinality of the set-- I mean, that was-- at least the proof, the first time I saw it, that was some crazy shit.

And this is also going to be some pretty crazy and clever shit, too. So this is what's referred to as Cantor's diagonalization argument or diagonal argument. And it's the following. So we're going to prove this by contradiction.

So we're going to assume that this is countable. So it can't be finite. There's infinitely many of the elements in here. So suppose that, in fact, it has the cardinality of the natural numbers. And we're going to arrive at a contradiction. So then there exists a function, which I'm going to call x, from natural numbers into 0, 1, which is bijective.

Now, so let me label this inequality star. For each n, we write x of n. So this is an element of 0, 1. So this is bijective. So every element in 0 gets mapped onto. So we write x of n to be in its decimal representation. I have two indices now-- one for the digit place and the one for which-- excuse me. At least I sheltered you from that sneeze by putting my hand over the microphone. Is the microphone still on? Yeah.

OK, so for each n, we write each x of n in its decimal representation, satisfying star. So each x in here can be written uniquely in a given decimal representation satisfying this inequality here.

And what we're going to do is we're going to come up with a real number in the interval 0, 1 which x does not map to, which therefore contradicts the fact that x is surjective. And bijective surjective means that everything has to get mapped onto. That's the onto part. Injective means it maps different things to different things.

So what's the idea of finding this element that doesn't get mapped onto? So let me write down just a few of these decimal expansions. And I may not want to do a fourth one.

And maybe I do. So this is, again, discussion. So I'm going to put this in brackets, meaning this is not part of the proof, but this is part of the idea of what's going on.

And where's the diagonal part? OK, so and this should go off in this direction . And the decimal expansion is marked off in that direction. And so somehow, this list, if it kept going in that direction, would have to contain every real number between 0 and 1.

And so what's the idea to come up with a real number between 0 and one that is not in this list is we go down the diagonal of the decimal expansions and we change each decimal to something else than what it is.

And so therefore-- and then I take my element y. So remember, we're trying to find a y not in this list. And then I would take y to be the real number that I get by changing each of these decimal digits.

So you can imagine if y somehow popped up-- let's say it popped up, since I don't want to write anymore, as x4, then somehow this number would have to be the digit for y, but I changed it. And therefore, it's not the digit of y. So that's it in a nutshell.

If you didn't get that, that's fine. Let me now write out the details. So it's a little bit simpler to imagine if, let's say, all of these were 0's and 1's. So instead of being base 10, now it's base 2. And so instead of-- if the digit was the 0, I've flipped it to 1. So you can imagine that if y did-- so and then I form y by flipping the digits. So if this was 0, 0, 1, 1, then y would start off 1, 1, 0, 0.

And therefore, if y appeared, say, fourth in line, it would have to be 1 and 0 in the same spot just by how I've constructed y by flipping it. And therefore, y can't be in that list. And therefore, this map cannot be surjective, which is our contradiction.

So let me make this more precise. Let ej-- this is going to be 1. If the j-th digit of the j-th x of j does not equal 1; 2, if the j-th digit of the j-th guy in that list, or x of j, does equal 1.

By part 1 of the theorem, previous theorem, there exists the unique y in 0, 1 without 0-- why? Because some of these digits are non-zero. It's always either 1 or 2-- such that y is equal to e minus 1, e minus 2, and so on.

So moreover, since all of these digits are either 1 or 2, they are certainly non-zero. So y is given by a decimal expansion where all of these things are non-zero.

So that means that for all n, a natural number, if I take a finite decimal expansion-- so I cut this off at e to the minus n-- since e to the e minus n plus 1 is either 1 or 2, that's going to be positive. So that's less than y. And this is always the case, that this is less than or equal to.

So because all these digits are positive, if I cut it off, if I truncate it, then that number is going to be less than y because I'm missing stuff out in the millionths place or whatever each time for each n. And therefore, this decimal expansion is the unique decimal representation from part 2 of the theorem that I stated.

Every element in 0, 1 has a unique representation, satisfying that inequality. Because all of these digits are positive, this, in fact, is a unique representation of this element y, which I've constructed by flipping digits from this map.

Now, since x is surjective, there exists an n, natural number-- this n has nothing to do with that n. Maybe I'll use a different letter, m-- such that y equals x of m. Then let's say I look at e to the minus m.

So then this says that the m-th digit, which should be giving me e to the minus m because this is equal to -- this, remember, is equal to 1 if dm 2 if dm equals 1. In particular, this does not equal dm. So I have proven that this number does not equal itself.

All right, I can never if-- I can never have this number, whatever it is, equal to this thing because I'm always changing it. If this thing is 1, then this number is 2. If this number is not 1, then this number is 1. And therefore, it's not equal to that. And this is a contradiction. Thus, it shows that this set is uncountable.

So I hope I explained that well enough. You can also look in the textbook for an explanation as well. And of course, you can ask me in office hours for more explanation. Let me fuel up real quick. OK, now, we're moving onto a new chapter. We're going to get to-- now, we're going to get to the analysis part.

So analysis as I've said before, is the study of limits. Real analysis is the study of limits that has to deal with real numbers. So let's get to our first notion of a limit. And that is the limit of a sequence of real numbers. So now, we're moving onto sequences and series.

So what is the precise definition of a sequence of real numbers? A sequence of real numbers is precisely a function x from the natural numbers into R, just a function. It doesn't have to be bijective, surjective, injective, anything. It's just a function from the natural numbers to R. So that's precisely what a sequence of real numbers is.

Now, we typically don't think of a sequence as a function. In fact, we don't even use that notation. We denote x of n by x sub n and the associated sequence by this curly brackets, n equals 1 to infinity. Or we might not even write the n equals 1 to infinity part or just start listing them.

All right, so let me quickly say that although-- so again, a sequence, unambiguously defined, is a function from the natural numbers to real. So don't confuse this with a set. A sequence is not a set. So think of a sequence, even though again it's defined as a function, as just an infinite listing of elements of R-- one element, the first element, the second element, the third element, and so on.

And they don't have to be different. So for example, 1, 1, 1, 1, that is a perfectly good sequence. In terms of what is the function, this is x of n equals 1 for all n. But again, I will not think of really a sequence as a function. Just as we never thought about functions as subsets of-- as certain subsets of the Cartesian product of two sets, you should really think of these as just a list of real numbers.

And when we refer to them, we'll either write them like that or with curly brackets around an expression for x sub n. So for example, if I were to write this, this means this is the sequence given by 1, 1/2, 1/3, 1/4, 1/5, and so on. OK.

So here's another definition. So a sequence is bounded if, in some sense, it doesn't run off to arbitrarily large values. So sequence x sub n is bounded if there exists some real number b bigger than or equal to 0 such that for all n in the natural numbers xn is less than or equal to b in absolute value.

So again, for example, if we take a look back at both of those two sequences-- 1, 1, 1, and the sequence 1 over n-- these are bounded. Why? Because every element in this-- I shouldn't say element, but every entry in this sequence is equal to 1. So it's bounded by 1. And every entry in this sequence is less than or equal to 1. So it's bounded by 1 in absolute value. It's also positive. So they're both bounded.

Another example is the sequence minus 1 to the n. So I'll write equals, although this shouldn't-- this doesn't mean anything. I'm just going to write it-- write it-- or start listing the entries-- minus 1, 1, minus 1, 1, minus 1, 1. This is also bounded because the absolute value of minus 1 to the n is 1. So that's bounded by 1 for all n.

So what's a non-example? The sequence n-- in other words, this is the sequence 1, 2, 3, 4, 5, and so on. Now, precisely why is it un-- so I should say this is bounded. And otherwise, I'll refer to x sub n as unbounded, unbounded.

So why is it unbounded? Because the entries are getting larger and larger in size. But if I were to ask you to prove this, you would have to show-- and this is also a good first exercise-- what does it mean to be unbounded. So I said this at one point, that if you come across a reasonably interesting definition, you should look up or try to come up with examples. And typically, when you do come up with examples, you should come up with non-examples.

And what that requires you to do is negate the definition. What does it mean for something to not be that? So even though I'm saying that if it's not bounded I call it unbounded, I haven't actually written a mathematical statement to this effect on what this means. So let me make this right here as a remark.

So a sequence is unbounded, so if it doesn't satisfy that definition. So we need to negate this definition. The definition says this is bounded if there exists a b bigger than or equal to 0 such that this holds for all n. Now, when I negate a there exists, that becomes a for all. And when I negate a for all, that becomes a there exists. And then I negate this condition.

So the sequence is unbounded if for all b bigger than or equal to 0 there exists a natural number in n such that xn in absolute value is bigger than b. Now, using this-- so this is-- if you like, you can take that as a second definition. Or it's not really a definition, since it's just the negation of the first definition.

Why is this set unbounded? Because of the Archimedean property. So now, I'm giving you a little short proof on why this sequence is unbounded. Let b be bigger than or equal to 0. By the Archimedean property, there exists a natural number n such that n is bigger than b, which is exactly what we wanted to prove because, in this case, x sub n is n, is unbounded.

OK, now, what does it mean for us to have a limit of a sequence? What does a limit of a sequence mean? What does it mean for a real number to be the limit of a sequence? What does it mean for a sequence to converge to something?

So the sequence x sub n converges to a real number x in R if the following condition is satisfied-- if for every epsilon positive, there exists a natural number capital M such that for all n bigger than or equal to capital M xn minus x absolute value is less than epsilon. Now, remember the absolute value is meant to be something like a distance.

So what does this statement say? This says if I give you a little bit of tolerance, epsilon, and you go far enough out in the sequence, then the distance between that entry in the sequence and x are very close to each other. Let me finish stating the rest of this definition and we'll do a little more discussion.

If a sequence converges, we say it's convergent. Otherwise, we say it's divergent. So all right, let me draw-- let me draw a little picture of what this is meant to be.

So the real numbers, it was this-- I mean, it is this unique ordered field with the least upper bound property. But when you think about the real numbers, think of it as you've always thought about it, as just the real number line.

So what does this definition mean? It means that a sequence converges to some x if for every epsilon I can do the following. If I look at-- if I give you an epsilon and you go on either side of x epsilon amount, then you should be able to find a capital M so that if you look at entries in your sequence x sub n for n bigger than or equal to M they're in this guy.

So then I should have x sub M there, x sub M plus 1, x sub M plus 2. And in general, x sub n should be in there, as long as n is bigger than or equal to M. So I should be able to do this for every epsilon. So for any amount of tolerance epsilon, you should be able to go far enough out in the sequence that all entries in that sequence are within this tolerance of x.

I mean, although this is a picture of what the definition means, how should you-- I mean, you've probably gotten plenty of experience with sequences when you were doing calculus. But what does this mean loosely? It means if I look close to x, some little interval containing x, then all of the sequence, all of the elements of the sequence should be eventually in this-- in this little interval.

Or another way to think about it is that as I go on in the sequence, the entries are getting closer and closer and closer to x Now, the way one makes these last two intuitive statements precise is via this definition. The closer and closer part is encapsulated in this for all epsilon part. And eventually getting closer and closer means for all n bigger than or equal to capital M. As long as I go far enough out in the sequence, they're getting close to x. So I hope that's pretty clear.

All right, so again, we have a definition. And it's reasonably interesting. So we should do examples, and then also negate it. But before I do either of those, I do want to just prove a very simple fact about convergent sequences, namely that if I have a sequence which converges to x, then x is the only thing that sequence can converge to. I cannot have a convergent sequence which converges to two different things.

Like I said, if you go by this intuition that eventually the x sub n's are getting closer and closer and closer to x, well, then there's no way they can be also getting closer to something other than x. So I just want to prove that real quick. And then we'll do examples and a negation of this definition.

So the following theorem-- and so I'm going to state first the theorem that has nothing to do with sequences, but is a nice way if you want to show two things are equal. You can show they're smaller than anything.

So the first theorem that I'm going to state is if x and y are in R and all epsilon positive x minus y is less than epsilon, then x equals y. This is not a surprising statement. If I have x and y and the distance between them is arbitrarily small, then they have to be the same thing.

So what's the proof? Suppose xy is in R. And for all epsilon positive, x minus y is less than epsilon. So now, I want to prove x equals y. So let's assume that this does not hold and arrive at a contradiction. So this is a short proof by contradiction.

Suppose x does not equal y. Then remember we proved that the absolute value of something is 0 if and only if that thing is equal to 0 and it's always non-negative. So x not equal to y means the absolute value of x minus y is less than-- is bigger than 0.

Then by this assumption, for epsilon equal to x minus y over 2, I get-- which implies if I just subtract that over, x minus y is less than 0. So 1/2 is less than 0 or-- and that's a very false statement. We've already proven that the absolute value is always non-negative.

So if I have two real numbers that are arbitrarily close to each other, then they have to be the same. I'm going to use this theorem to prove that-- I don't know why I put a 1 there-- why limits-- and I have not said that's what we call x-- but why a convergent sequence can only converge to one thing.

So here is the statement of the second theorem. If x sub n is a sequence and it converges to x and it also converges to y, then x equals y. So a convergent sequence can only converge to one thing.

And again, like I said, there should be clear because what's happening is if I have two things that are not equal and x sub n converges to x, that means as long as I go far enough out in the sequence, they're supposed to be here in this small interval around x.

Well, if they're supposed to be converging to y, if I go far enough out, they're also supposed to be over here as long as I go far enough out. But you can't be in two places at once ever. And so that's why x must equal y. Although, I suppose that now we're doing all the classes online, some are recorded. So you can be two places at once, but at least not here.

OK, so we're going to use this theorem to prove this statement. Suppose-- converges to x and y. And I want to verify that for all epsilon positive the absolute value of x minus y is less than or equal to epsilon.

Now, remember if I'm proving something for all epsilon, I have to-- that means I do it for arbitrary epsilon. You get points just for setting let epsilon be positive. OK, now, I want to show the absolute value of x minus y is less than or equal to epsilon.

And we're basically going to do the argument that I just erased. So since x sub n converges to x, that implies that there exists a natural number M1 such that for all n bigger than or equal to M1, x sub n minus x is less than epsilon over 2. Why the epsilon over 2? You'll see, magic happens.

And since x sub n converges to y, thus this implies that there exists an M2 natural number such that for all n bigger than or equal to M2-- so again, this is the part that says as long as I go far enough out s sub n has to be close to x. Now, this is the part that says as long as I go far enough out x sub n has to be close to y-- so that for all n bigger than or equal to M2 x sub n minus y is less than epsilon over 2.

Then since the sum M1 plus-- these are both natural numbers. This is bigger than or equal to M1 and M2. This implies-- now, I'm going to use these two inequalities and the triangle inequality. Then if I look at the absolute value of x minus y, this is equal to the absolute value of x minus x sub n plus x sub n minus-- so let me-- not x sub n, but x sub M1 plus M2 plus x sub M1 plus M2.

Now, by the triangle inequality, this is less than or equal to x M2 plus M1 plus M2 minus y. Now, this is less than epsilon over 2 because M1 plus M2 is bigger than or equal to M1. And for the same reason, since plus M1 plus M2 is bigger than or equal to M2, this is less than epsilon over 2.

Ah, now you see why I divided both of those by 2 because when I add them up, I get epsilon. So I've shown that for every epsilon positive, the absolute value of x minus y is less than epsilon. And therefore, by the theorem that I proved a minute ago, x equals y.

Why did I state that theorem? Because I want to use notation. And I want this notation to be already apparent that it's consistent. And I'm going to use terminology. So I mean, this is not really a new definition. It's just some terminology.

We call x the limit of the sequence. Now, the fact that I've shown only that there's only one guy that a sequence can converge to is why I get to use the word "the--" so not we call x a limit of x sub n. It's the limit because there's only one, if one exists-- and write x equals limit as n goes to infinity of x sub n.

OK, so let's pause on negating the definition of what it means for a sequence to be convergent. And let's do a couple of examples of convergent sequences, proving that they are convergent sequences and computing what they-- and at least proving that these sequences do converge to the limit that I'm telling you.

So for example, let's look at the constant sequence x sub n equals 1 for all N. So that's the sequence 1, 1, 1, and so on. Then I'm just going to write 1, not x sub n. The limit as n goes to infinity of 1 equals 1. This proof is not very enlightening, but I'll show you how it goes.

So you're supposed to show-- what does this mean? This means for every epsilon positive, there exist a natural number capital M such that if I'm bigger than capital M, this minus the limit should be less than epsilon. So let epsilon be positive. I have to now come up with a capital M so that x sub n minus x is less than epsilon whenever little n is bigger than or equal to capital M. So I'm going to choose capital M to be 1. I can do that for this sequence.

Then if n is bigger than or equal to M and I look at the n-th entry in the sequence, well, this is just 1 minus 1 equals 0. And that's less than epsilon. So not very enlightening because capital M equals 1 is good enough. So let's do something a little more.

So how about the limit as n goes to infinity of 1 over n equals 0? So let's do a proof of this. Again, I'm supposed to be proving this epsilon statement in the definition in order to be able to state this, which is for every epsilon statement. So I start the proof off with let epsilon be positive.

Now, I have to find-- I have to tell you how to choose capital M to ensure that little n bigger than or equal to capital M implies x sub n minus 0, in this case, is less than epsilon. So I will say choose capital M natural number such that m is bigger than 1 over epsilon. In fact, I'm not going to write it that way.

I'm going to write it this way-- 1 over M is less than epsilon. And Why Can I do this? Why can I find such a natural number that satisfies this? And you can't just insert statements into your proof-- choose capital M to satisfy the unicorn property-- and not clarify why you can choose such a natural number that satisfies the unicorn property.

I can do this by the Archimedean property. This is equivalently stating that I can find a natural number M so that n is bigger than 1 over epsilon, what I had written there originally. So let me write it this way-- such an M exists by the Archimedean property of the natural numbers.

I'll say that a few times if I have to or to clarify why certain natural numbers satisfying certain properties exist. And then at one point, I'll just stop because it should be clear that I've done it enough times you know how or why something does exist.

OK, so I choose a natural number and a natural number M so that I have this. Now, I need to show you that this number works. So let n be bigger than capital M. Then if I look at x sub n minus my proposed limit, which is 0-- so I look at 1 minus n 0-- this is equal to just 1 over n. Since little n is bigger than or equal to capital M, this is less than or equal to capital M, which is less than epsilon.

And therefore, I've proven what I wanted to prove. So how do these-- if I just give you a sequence and I ask you to prove that-- so this is some discussion. How do you come up with these proofs? How should the proof typically look? If I want to prove-- equals L or x, what's the proof of that look like?

It should always-- so the proof should be let epsilon be positive. And then maybe you need to do some explaining. And then you'll say choose M so that something. And that such an M exists probably will be explained there or in the same sentence.

And then the next part of the proof should be showing that this capital M works. So then if n is bigger than or equal to capital M, then you should show that it works by looking at x sub n minus x and doing a calculation or inequalities and arriving at this being less than epsilon, which is what you're wanting to prove.

You're wanting, in the end, to verify this definition, which says for every epsilon, you can find a capital M so that I get this. So in a proof, you are carrying that out. You're saying for epsilon positive, I choose M so that this happens.

Now, how do you-- how do you come up with such a capital M? So again, this is just discussion. And maybe we won't get to any more examples. I'll just finish this discussion, and then we'll call it a day. How do you come up with such a capital M? How did I come up with M? Why did I choose M in this way?

In the end, you can see it here, is that I chose it because if I fiddle around with the thing I want to bound, I ended up with 1 over n, which I can control and make it smaller than epsilon. So typically, how do you find capital M? So this is a discussion, I guess, within a discussion.

Typically, you'll take x sub n minus x, your proposed limit. Maybe I give you a sequence explicitly. Or maybe it's a expression involving some other sequences. And you fiddle around with it. Maybe you write it differently, or add and subtract something, or multiply it by 1 in a fancy way. And you'll get some expression typically involving something involving n.

And as long as this expression involving n is simple enough, then you choose capital M so that this thing is less than epsilon. Now, simple enough, for example, 1 over n, I can choose capital M so that this is less than epsilon by the Archimedean property.

If, for example-- and that's basically 1 over n, or 5 over n, or something like that, that exhausts all the simplest ones for now. But if, for example, you ended up with 1 over n squared minus 3n plus 100, I would not say this is a simple enough expression to be able to say choose capital M so that this is less than epsilon without more explanation than what I just gave here.

And we'll do some more examples next time to flesh this out a little bit more. All right, so we'll stop there.