Exercises given with a numbering are from *Basic Analysis: Introduction to Real Analysis (Vol I)* by J. Lebl.

Reading Sections 1.2, 1.3, 1.4, 1.5, 2.1

Exercises

1. Suppose $x, y \in \mathbb{R}$ and $x < y$. Prove that there exists $i \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < i < y$.
2. Let $E \subset (0, 1)$ be the set of all real numbers with decimal representation using only the digits 1 and 2:

$$E := \{x \in (0,1) : \forall j \in \mathbb{N}, \exists d_{-j} \in \{1,2\} \text{ such that } x = 0.d_{-1}d_{-2}\ldots\}$$

Prove that $|E| = |\wp(\mathbb{N})|$. *Hint*: Consider the function $f : E \to \wp(\mathbb{N})$ such that if $x \in E, x = 0.d_{-1}d_{-2}\ldots$,

$$f(x) = \{j \in \mathbb{N} : d_{-j} = 2\}.$$

3. (a) Let A and B be two disjoint, countably infinite sets. Prove that $A \cup B$ is countably infinite.
   (b) Prove that the set of irrational numbers, $\mathbb{R}\backslash\mathbb{Q}$, is uncountable. You may use the facts discussed in the lectures that $\mathbb{R}\backslash\mathbb{Q}$ is infinite and $\mathbb{R}$ is uncountable without proof.
4. Let A be a subset of $\mathbb{R}$ which is bounded above, and let a_0 be an upper bound for A. Prove that $a_0 = \sup A$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $a_0 - \epsilon < a$.
5. We say a set $U \subset \mathbb{R}$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset U.$$

Since the definition is vacuous for $U = \varnothing$, it follows that the empty set is open. It is also clear from the definition that $U = \mathbb{R}$ is open.

   (a) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that the sets $(-\infty, a), (a, b)$, and (b, ∞) are open.
   (b) Let Λ be a set (not necessarily a subset of $\mathbb{R}$), and for each $\lambda \in \Lambda$, let $U_\lambda \subset \mathbb{R}$. Prove that if U_λ is open for all $\lambda \in \Lambda$ then the set

$$\bigcup_{\lambda\in\Lambda} U_\lambda = \{x \in \mathbb{R} : \exists \lambda \in \Lambda \text{ such that } x \in U_\lambda\}$$

   is open.
   (c) Let $n \in \mathbb{N}$, and let $U_1, \ldots U_n \subset \mathbb{R}$. Prove that if $U_1, \ldots, U_n$ are open then the set

$$\bigcap_{m=1}^{n} U_m = \{x \in \mathbb{R} : x \in U_m \text{ for all } m = 1, \ldots, n\}$$

   is an open set.

(d) Is the set of rationals $\mathbb{Q} \subset \mathbb{R}$ open? Provide a proof to substantiate your claim.

6. Prove that

$$\lim_{n\to\infty} \frac{1}{20n^2 + 20n + 2020} = 0.$$