18.100A: Complete Lecture Notes

Lecture 12:
The Ratio, Root, and Alternating Series Tests

We continue our study of convergence tests.

Theorem 1 (Ratio test)
Suppose $x_n \neq 0$ for all n and

$$L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists. Then,

1. if $L < 1$ then $\sum x_n$ converges absolutely.
2. if $L > 1$ then $\sum x_n$ diverges.

Proof: We will first prove the second part of this theorem.

2) Suppose $L > 1$ and $\alpha \in (1, L)$. Then, there exists $M_0 \in \mathbb{N}$ such that for all $N \geq M_0$, $\frac{|x_{n+1}|}{|x_n|} \geq \alpha \geq 1$. Thus, for all $n \geq M_0$,

$$|x_{n+1}| \geq |x_n| \implies \lim_{n\to\infty} |x_n| \neq 0.$$

Therefore, $\sum x_n$ diverges.

1) Now suppose that $L < 1$. Let $\alpha \in (L, 1)$. Then, there exists $M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$, $\frac{|x_{n+1}|}{|x_n|} < \alpha$. Therefore,$\forall n \geq M_0$, $|x_{n+1}| \leq \alpha|x_n|$. In other words, for all $n \geq M_0$,

$$|x_n| \leq \alpha|x_{n-1}| \leq \alpha^2|x_{n-2}| \leq \cdots \leq \alpha^{n-M_0}|x_{M_0}|.$$

Let $m \in \mathbb{N}$. Then,

$$\begin{aligned}
\sum_{n=1}^{m} |x_n| &= \sum_{n=1}^{M_0-1} |x_n| + \sum_{n=M_0}^{m} |x_n| \\
&\leq \sum_{n=1}^{M_0-1} |x_n| + |x_{M_0}| \sum_{n=M_0}^{m} \alpha^{n-M_0} \\
&\leq \sum_{n=1}^{M_0-1} |x_n| + |x_{M_0}| \sum_{\ell=0}^{\infty} \alpha^{\ell} \\
&= \sum_{n=1}^{M_0-1} |x_n| + \frac{|x_{M_0}|}{1-\alpha}.
\end{aligned}$$

Therefore, $\{\sum_{n=1}^{m} |x_n|\}_{m=1}^{\infty}$ is bounded, and thus $\sum |x_n|$ converges. Hence, x_n is absolutely convergent.

□

Let's consider two examples where we can use the Ratio test.

Example 2

Show the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ converges absolutely.

Proof: Notice

$$\left|\frac{(-1)^n}{n^2+1}\right| \leq \frac{1}{n^2+1} < \frac{1}{n^2},$$

and hence

$$\lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}}{(n+1)^2+1}}{\frac{(-1)^n}{n^2+1}}\right| < \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$$

■

Example 3

Show that $\forall x \in \mathbb{R}$, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely.

Proof: This immediately follows from the Ratio test, noting that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

■

Remark 4. *As seen above, the Ratio test can be really helpful to use when we have a* $(-1)^n$ *or a factorial in the argument. Also note that if* $L = 1$ *then we the test doesn't apply.*

Theorem 5 (Root test)

Let $\sum x_n$ be a series and suppose that

$$L = \lim_{n\to\infty} |x_n|^{1/n}$$

exists. Then,

1. if $L < 1$ then $\sum x_n$ converges absolutely.
2. if $L > 1$ then $\sum x_n$ diverges.

Proof:

1. Suppose $L < 1$. Let $L < r < 1$. Then, since $|x_n|^{1/n} \to L$, $\exists M \in \mathbb{N}$ such that $\forall n \geq M$, $|x_n|^{1/n} < r$. Therefore, for all $n \geq M$, $|x_n| \leq r^n$. Thus, for all $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{m} |x_n| &= \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{m} |x_n| \\ &\leq \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{m} r^n \\ &\leq \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} r^n \\ &= \sum_{n=1}^{M-1} |x_n| + \frac{r^M}{1-r}.\end{aligned}$$

Thus, $\{\sum_{n=1}^{m} |x_n|\}_{m=1}^{\infty}$ is bounded, and thus $\sum |x_n|$ converges.

2. Suppose $L > 1$. Then, since $|x_n|^{1/n} \to L > 1$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, $|x_n|^{1/n} > 1$. In other words, for all $n \geq M$, $|x_n| > 1$. Therefore, $\lim_{n\to\infty} x_n \neq 0$, and thus $\sum x_n$ diverges.

□

Remark 6. *Again, note that if $L = 1$ then the test doesn't apply.*

Theorem 7 (Alternating Series test)
Let $\{x_n\}$ be a monotone decreasing sequence such that $x_n \to 0$. Then, $\sum(-1)^n x_n$ converges.

Proof: Let $s_m = \sum_{n=1}^m (-1)^n x_n$. Then,

$$\begin{aligned} s_{2k} &= \sum_{n=1}^{2k} (-1)^n x_n \\ &= (x_2 - x_1) + (x_4 - x_3) + \cdots + (x_{2k} - x_{2k-1}) \\ &\geq (x_2 - x_1) + \cdots + (x_{2k} - x_{2k-1}) + (x_{2k+2} - x_{2k+1}) \\ &= s_{2(k+1)} \end{aligned}$$

as $\{x_n\}$ is a monotone decreasing sequence. Thus, $\{s_{2k}\}_{k=1}^\infty$ is monotone decreasing. Furthermore,

$$s_{2k} = -x_1 + (x_2 - x_3) + (x_4 - x_5) + \cdots + (x_{2k-2} - x_{2k-1}) + x_{2k} \geq -x_1.$$

In other words, $\{s_{2k}\}$ is a bounded below monotone decreasing sequence. Thus, $\{s_{2k}\}_{k=1}^\infty$ converges. Let $s = \lim_{k\to\infty} s_{2k}$. We now prove $\{s_m\}_{m=1}^\infty$ converges to s.

Let $\epsilon > 0$. Since $s_{2k} \to s$, $\exists M_0 \in \mathbb{N}$ such that for all $k \geq M_0$,

$$|s_{2k} - s| < \frac{\epsilon}{2}.$$

Since $x_n \to 0$, $\exists M_1 \in \mathbb{N}$ such that $\forall n \geq M_1$,

$$|x_n| < \frac{\epsilon}{2}.$$

Choose $M = \max\{2M_+0 + 1, M_1\}$. Suppose $m \geq M$. If m is even, then $\frac{m}{2} \geq M_0 + 1/2 \geq M_0$. Therefore,

$$|s_m - s| = |s_{2\cdot\frac{m}{2}} - s| < \frac{\epsilon}{2} < \epsilon.$$

If m is odd, let $k = \frac{m-1}{2}$ so $m = 2k + 1$. Then, $m \geq M \implies k \geq M_0$ and $m \geq M_1$. Then,

$$\begin{aligned} |s_m - s| &= |s_{m-1} + x_m - s| \\ &\leq |s_{2k} - s + x_m| \\ &\leq |s_{2k} - s| + |x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, $s_m \to s$, and thus $\sum(-1)^n x_n$ converges. □

Corollary 8
We already showed that $\sum \frac{(-1)^n}{n}$ does not absolutely converge. However, $\sum \frac{(-1)^n}{n}$ converges.

Proof: This follows immediately from the Alternating Series test.

Theorem 9

Suppose $\sum x_n$ converges absolutely and $\sum x_n = x$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijective function. Then, $\sum x_{\sigma(n)}$ is absolutely convergent and $\sum x_{\sigma(n)} = x$. In other words, absolute convergence implies if we rearrange the sequence the new series will still converge to the same value of the original series.

Proof: We first show $\sum |x_{\sigma(n)}|$ converges, which is equivalent to showing the partial sums $\sum_{n=1}^{m} |x_{\sigma(n)}|$ is bounded. Since $\sum x_n$ converges, $\exists B \geq 0$ such that for all $\ell \in \mathbb{N}$,

$$\sum_{n=1}^{\ell} |x_n| \leq B.$$

Let $m \in \mathbb{N}$. Then, $\sigma(\{1, \ldots, m\})$ is a finite subset of $\mathbb{N}$. Thus, there exists an $\ell \in \mathbb{N}$ such that

$$\sigma(\{1, \ldots, m\}) \subset \{1, \ldots, \ell\}.$$

Thus,

$$\sum_{n=1}^{m} |x_{\sigma(n)}| = \sum_{n \in \sigma(\{1, \ldots, m\})} |x_n| \leq \sum_{n=1}^{\ell} |x_n| \leq B.$$

Therefore, $\sum |x_{\sigma(n)}|$ converges. Let $x = \sum_{n=1}^{\infty} x_n$, and let $\epsilon > 0$. Then, $\exists M_0 \in \mathbb{N}$ such that $\forall m \geq M_0$,

$$\left| \sum_{n=1}^{m} x_n - x \right| < \frac{\epsilon}{2}.$$

Since $\sum |x_n|$ converges, $\exists M_1 \in \mathbb{N}$ such that for all $\ell > m \geq M_1$,

$$\sum_{n=m+1}^{\ell} |x_n| < \frac{\epsilon}{2}.$$

Let $M_2 = \max\{M_0, M_1\}$. Then, $\forall \ell > m \geq M_2$,

$$\left| \sum_{n=1}^{m} x_n - x \right| < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{n=m+1}^{\ell} |x_n| < \frac{\epsilon}{2}.$$

Since $\sigma^{-1}(\{1, \ldots, M_2\})$ is a finite set, $\exists M_3 \in \mathbb{N}$ such that

$$\{1, \ldots, M_2\} \subset \sigma(\{1, \ldots, M_3\}).$$

Choose $M = M_3$. Thus, if $m' \geq M$,

$$\begin{aligned}
\left|\sum_{n'=1}^{m'} x_{\sigma(n')} - x\right| &= \left|\sum_{n\in\sigma(\{1,\ldots,m'\})} x_n - x\right| \\
&= \left|\sum_{n=1}^{M} x_n - x + \sum_{n\in\sigma(\{1,\ldots,m'\})\setminus\{1,\ldots,M\}} x_n\right| \\
&\leq \left|\sum_{n=1}^{M} x_n - x\right| + \sum_{n=M+1}^{\max\sigma(\{1,\ldots,m'\})} |x_n| \\
&\leq \left|\sum_{n=1}^{M} x_n - x\right| + \sum_{n=M+1}^{\ell} |x_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

$\square$