18.100A: Complete Lecture Notes

Lecture 13:
Limits of Functions

Continuous Functions

Remark 1. *Continuous functions are those functions where tolerable changes to outputs accompany sufficiently small differences of inputs.*

Limits of Functions

Definition 2 (Cluster Point)
Let $S \subset \mathbb{R}$. $x \in \mathbb{R}$ is a cluster point of S if $\forall \delta > 0$, $(x - \delta, x + \delta) \cap S \setminus \{x\} \neq \emptyset$.

Let's look at some examples.

1. $S = \{1/n \mid n \in \mathbb{N}\}$. Here, 0 is a clusterpoint of S.
2. $S = (0, 1)$. The set of cluster points of S is $[0, 1]$.
3. $S = \mathbb{Q}$. The set of cluster points of S is $\mathbb{R}$.
4. $S = \{0\}$. There are no cluster points of S.
5. $S = \mathbb{Z}$. There are no cluster points of S.

Theorem 3
Let $S \subset \mathbb{R}$. Then, x is a cluster point of S if and only if there exists a sequence $\{x_n\}$ of elements in $S \setminus \{x\}$ such that $x_n \to x$.

Definition 4 (Function Convergence)
Let $S \subset \mathbb{R}$, let c be a cluster point of S, and $f : S \to \mathbb{R}$. We say that $f(x)$ converges to $L \in \mathbb{R}$ at c if $\forall \epsilon > 0$ $\exists \delta > 0$ such that if $x \in S$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Notation 5
Notationally, we may write $f(x) \to L$ as $x \to c$, or $\lim_{x \to c} f(x) = L$.

Theorem 6
Let c be a cluster point of $S \subset \mathbb{R}$, and let $f : S \to \mathbb{R}$. If $f(x) \to L_1$ and $f(x) \to L_2$ as $x \to c$, then $L_1 = L_2$.

Proof: We will show $\forall \epsilon > 0$, $|L_1 - L_2| < \epsilon$. Let $\epsilon > 0$. Then, since $f(x) \to L_1$ and $f(x) \to L_2$, $\exists \delta_1$ such that if $x \in S$ and $0 < |x - c| < \delta_1$ then

$$|f(x) - L_1| < \epsilon/2$$

and $\exists \delta_2 > 0$ such that if $x \in S$ and $0 < |x - c| < \delta_2$, then

$$|f(x) - L_2| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, since c is a cluster point of S, $\exists x_0 \in S$ such that

$$0 < |x_0 - c| < \delta \implies |L_1 - L_2| = |L_1 - f(x_0) + f(x_0) + L_2| \leq |L_1 - f(x_0)| + |f(x_0) - L_2| < \epsilon.$$

□

Let's see some examples of limits of functions.

Example 7

Let $f(x) = ax + b$. Then, for all $c \in \mathbb{R}$, $\lim_{x \to c} f(x) = ac + b$.

Proof: Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{1+|a|}$. Then, if $x \in \mathbb{R}$ and $0 < |x - c| < \delta$, then

$$\begin{aligned} |f(x) - (ac + b)| &= |a(x - c)| \\ &= |a||x - c| \\ &< |a|\delta \\ &= \frac{|a|}{1 + |a|}\epsilon < \epsilon. \end{aligned}$$

■

Example 8

Let $f(x) = \sqrt{x}$. Then, $\forall c > 0$, $\lim_{x \to c} f(x) = \sqrt{c}$.

Proof: Let $\epsilon > 0$. Choose $\delta = \epsilon\sqrt{c}$. Then, if $x > 0$ and $0 < |x - c| < \delta$, then

$$\begin{aligned} |f(x) - \sqrt{c}| &= |\sqrt{x} - \sqrt{c}| \\ &= \left| \frac{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})}{\sqrt{x} + \sqrt{c}} \right| \\ &= \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \\ &\leq \frac{|x - c|}{\sqrt{c}} \\ &< \frac{\delta}{\sqrt{c}} = \epsilon. \end{aligned}$$

■

Example 9

Let $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$. Then, $\lim_{x \to 0} f(x) = 1$. Notably, $\lim_{x \to 0} f(x) \neq f(0)$!

Proof: Let $\epsilon > 0$ and choose $\delta = 1$. Then, if $0 < |x - 0| < 1$ then $x \neq 0 \implies$

$$|f(x) - 1| = |1 - 1| = 0 < \epsilon.$$

■

Question 10. *How do limits of functions relate to limits of sequences?*

Theorem 11

Let $S \subset \mathbb{R}$, c a cluster point of S, and let $f : S \to \mathbb{R}$. Then, the following are equivalent:

1. $\lim_{x \to c} f(x) = L$ and
2. for every sequence $\{x_n\}$ in $S \setminus \{c\}$ such that $x_n \to c$, we have $f(x_n) = L$.

Proof: (1. $\implies$ 2.): Suppose $\lim_{x \to c} f(x) = L$. Let $\{x_n\}$ be a sequence in $S \setminus \{c\}$ such that $x_n \to c$. We want to show that $f(x_n) \to L$. Let $\epsilon > 0$. Given $\lim_{x \to c} f(x_n) = L$, $\exists \delta > 0$ such that if $x \in S$ and $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$. Since $x_n \to c$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$, $0 < |x_n - c| < \delta$.

Choose $M = M_0$. Then, $\forall n \geq M$, if $0 < |x_n - c| < \delta$ then $|f(x_n) - L| < \epsilon$. Thus, $f(x_n) \to L$.

(2. $\impliedby$ 1.): Suppose 2. holds, and assume for the sake of contradiction that 1) is false. Then, $\exists \epsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x \in S$ such that

$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - L| \geq \epsilon_0.$$

Then, $\forall n \in \mathbb{N}$, $\exists x_n \in S$ such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon_0$. By the Squeeze Theorem applied to

$$0 < |x_n - c| < \frac{1}{n},$$

$x_n \to c$. Then, by 2.,

$$0 = \lim_{n \to \infty} |f(x_n) - L| \geq \epsilon_0$$

which is a contradiction. □