18.100A: Complete Lecture Notes

Lecture 14:
Limits of Functions in Terms of Sequences and Continuity

Theorem 1
For all $c \in \mathbb{R}$, $\lim_{x\to c} x^2 = c^2$.

Proof: Let $\{x_n\}$ be a sequence in $\mathbb{R} \setminus \{c\}$ such that $x_n \to c$. Then, $x_n^2 \to c^2$ by a theorem shown in Lecture 8. Thus,

$$\lim_{x\to c} x^2 = c^2.$$

□

Theorem 2
We show that

1. $\lim_{x\to 0} \sin(1/x)$ does not exist, and
2. $\lim_{x\to 0} x\sin(1/x) = 0$.

Proof:

1. Let $x_n = \frac{2}{(2n-1)\pi}$. Then, $x_n \neq 0$, and $x_n \to 0$. But,

$$\sin(1/x_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n+1}$$

for all n. However, this sequence does not converge (i.e. the limit does not exist).

2. Suppose $x_n \neq 0$ and $x_n \to 0$. Then,

$$0 \le |x_n \sin(1/x_n)| = |x_n||\sin(1/x_n)| \le |x_n|.$$

By the Squeeze Theorem, $\lim_{n\to\infty} |x_n \sin(1/x_n)| = 0$. □

We can use the 'sequential limit' characterization to prove analogs of previous theorems for limits of sequences.

Theorem 3
Let $S \subset \mathbb{R}$, c a cluster point of S, and $f, g : S \to \mathbb{R}$. Suppose $\forall x \in S$, $f(x) \le g(x)$ and $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then,

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

Proof: Let $L_1 = \lim_{x\to c} f(x)$ and $L_2 = \lim_{x\to c} g(x)$. Let $\{x_n\}$ be a sequence in $S \setminus \{c\}$ such that $x_n \to c$. Then, $\forall n \in \mathbb{N}$, $f(x_n) \le g(x_n)$. Therefore,

$$L_1 = \lim_{n\to\infty} f(x_n) \le \lim_{n\to\infty} g(x_n) = L_2.$$

□

Similarly, we have analogs of the Squeeze Theorem, limits of algebraic operations, and limits of absolute values. You may read the end of Section 3.1.3 **[L]** for this.

Definition 4
Let $S \subset \mathbb{R}$ and suppose c is a cluster point of $S \cap (-\infty, c)$. Then, we say $f(x)$ converges to L as $x \to c^-$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $x \in S$ and $c - \delta < x < c$ then $|f(x) - L| < \epsilon$.

Notation 5
This is denoted $L = \lim_{x \to c^-} f(x)$.

Definition 6
Similarly, let $S \subset \mathbb{R}$ and suppose c is a cluster point of $S \cap (c, \infty)$. Then, we say $f(x)$ converges to L as $x \to c^+$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $x \in S$ and $c < x < c + \delta$ then $|f(x) - L| < \epsilon$.

Notation 7
This is denoted $L = \lim_{x \to c^+} f(x)$.

Example 8
Let $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$. Then,

$$\lim_{x \to 0^-} f(x) = 0 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = 1,$$

even though $f(0)$ is undefined.

Theorem 9
Let $S \subset \mathbb{R}$ and let c be a cluster point of $S \cap (-\infty, c)$ and $S \cap (c, \infty)$. Then, c is a cluster point of S. Moreover,

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

Continuous Functions

As we have seen, limits do not care about $f(x)$ when $x = c$. Continuity is a condition that connects $\lim_{x \to c} f(x)$ with $f(c)$.

Definition 10 (Continuous Functions)
Let $S \subset \mathbb{R}$ and let $c \in S$. We say f is continuous at c if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $x \in S$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$ We say f is continuous on U for $U \subset S$ if f is continuous at every point in U.

Example 11
$f(x) = ax + b$ is continuous on $\mathbb{R}$.

Proof: Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{1+|a|}$. If $|x-c| < \delta$, then

$$\begin{aligned} |f(x) - f(c)| &= |ax + b - (ac + b)| \\ &= |a||x - c| \\ &< |a|\delta \\ &= \frac{|a|}{1+|a|}\epsilon < \epsilon. \end{aligned}$$

□

Example 12

Show that $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$ is *not* continuous at $c = 0$.

First we write the negation of the definition of continuity.

Negation 13 (Not Continuous)

f is not continuous at c if $\exists \epsilon_0$ such that for all $\delta > 0$, $\exists x \in S$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.

Proof: Choose $\epsilon_0 = 1$ and let $\delta > 0$. Then, $x = \frac{\delta}{2}$ satisfies $|x - 0| < \delta$ and

$$|f(x) - f(0)| = |2 - 1| \geq 1 = \epsilon_0.$$

■

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