18.100A: Complete Lecture Notes

Lecture 20:
Taylor's Theorem and the Definition of Riemann Sums

Taylor's Theorem

Remark 1. *Taylor's theorem is essentially the Mean Value Theorem for higher order derivatives.*

Definition 2 (n-times Differentiable)
We say $f : I \to \mathbb{R}$ is n-times differentiable on $J \subset I$ if $f', f'', \ldots, f^{(n)}$ exist at every point in J.

Notation 3
We denote the n-th derivative of f as $f^{(n)}$ (as used above).

Theorem 4 (Taylor)
Suppose $f : [a, b] \to \mathbb{R}$ is continuous and has n continuous derivatives on $[a, b]$ such that $f^{(n+1)}$ exists on (a, b). Given $x_0, x \in [a, b]$, there exists a $c \in (x_0, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Denote the large sum as $P_n(x)$ and the last term with $R_n(x)$.

Definition 5
$P_n(x)$ is the n-th order Taylor polynomial for f at x_0. $R_n(x)$ is the n-th order remainder term.

We will essentially apply the Mean Value Theorem $n + 1$ times to prove Taylor's theorem.

Proof: Let $x, x_0 \in [a, b]$. If $x = x_0$ then any c will satisfy the theorem. So, suppose $x \neq x_0$. Let $M_{x,x_0} = \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}$. Hence,

$$f(x) = P_n(x) + M_{x,x_0}(x - x_0)^{n+1}.$$

Now, for $0 \leq k \leq n$,

$$f^k(x_0) = P_n^{(k)}(x_0).$$

Let $g(s) = f(s) - P_n(s) - M_{x,x_0}(s - x_0)^{n+1}$ (notably, $n + 1$-times differentiable. Then,

$$\begin{aligned}
g(x_0) &= f(x_0) - P_n(x_0) - M_{x,x_0}(x_0 - x_0)^{n+1} = 0 \\
g'(x_0) &= f'(x_0) - P_n'(x_0) - M_{x,x_0}(n+1)(x_0 - x_0)^n = 0 \\
&\vdots \\
g^{(n)}(x_0) &= f^{(n)}(x_0) - P_n^{(n)}(x_0) - M_{x,x_0}(n+1)!(x_0 - x_0) = 0.
\end{aligned}$$

Now, notice that $g(x) = 0$ and $g(x_0) = 0$. By the MVT, there exists an $x_1 \in (x_0, x)$ such that $g'(x_1) = 0$. Thus, $g'(x_0) = 0$ and $g'(x_1) = 0$. Therefore, $\exists x_2 \in (x_0, x_1)$ such that $g''(x_2) = 0$. Continuing, we analogously find x_n between x_0 and x_{n-1} such that $g^{(n)}(x_n) = 0$. Then, finally, $g^{(n)}(x_0) = 0$ and $g^{(n)}(x_n) = 0$ implies $\exists c \in (x_0, x_n)$ (and thus between x_0 and x) such that

$$g^{(n+1)}(c) = 0.$$

We may compute

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}s^{(n+1)}} M_{x,x_0}(s - x_0)^{n+1} = M_{x,x_0}(n+1)!.$$

Furthermore, $P_n^{(n+1)}(c) = 0$ since P_n is a polynomial of degree n. Thereforee,

$$0 = g^{(n+1)}(c) = f^{(n+1)}(c) - M_{x,x_0}(n+1)!,$$

which implies $M_{x,x_0} = \frac{f^{(n+1)!}(c)}{(n+1)!}$ and thus

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

□

Theorem 6 (Second Derivative Test)

Suppose $f : (a, b) \to \mathbb{R}$ has two continuous derivatives. If $x_0 \in (a, b)$ such that $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a strict relative minimum at x_0.

Proof: Since f'' is continuous at x_0 and

$$\lim_{c \to x_0} f''(c) = f''(x_0) > 0,$$

we have that $\exists \delta > 0$ such that for all $c \in (x_0 - \delta, x_0 + \delta)$, $f''(c) > 0$. Let $x \in (x_0 - \delta, x_0 + \delta)$ (as you will show in your homework). Then, by Taylor's theorem, $\exists c$ between x and x_0 (hence $c \in (x_0 - \delta, x_0 + \delta)$) such that

$$f(x) = f(x_0) + \frac{f''(c)}{2}(x - x_0)^2 \geq f(x_0),$$

with $f(x) > f(x_0)$ if $x \neq x_0$. □

The Riemann Integral

Remark 7. *Riemann integration is the first rigorous theory of 'area' that agrees with experience (areas of rectangles, triangles, circles), and it is the inverse of differentiation. However, it is not a complete theory of area (see Lebesgue integration).*

The Riemann Integral

Definition 8

We define the set

$$C([a, b]) := \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

Definition 9 (Partition)
A partition $\underline{x}$ of $[a, b]$ is a finite set

$$\underline{x} = \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

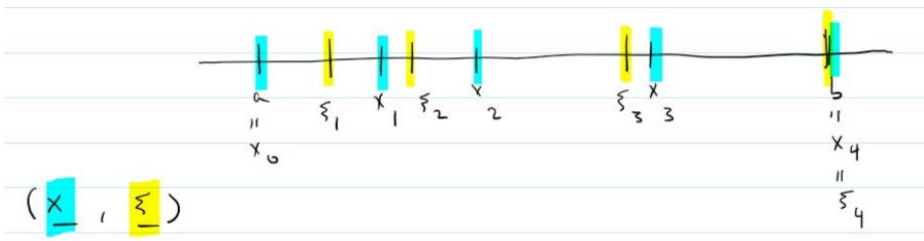
The norm of $\underline{x}$, denoted $\|\underline{x}\|$, is the number

$$\|\underline{x}\| := \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}.$$

Definition 10 (Tag)
If $\underline{x}$ is a partition, a tag of $\underline{x}$ is a finite set $\underline{\xi} = \{\xi_1, \ldots, \xi_n\}$ such that

$$a = x_0 \leq \xi_1 \leq x_1 \leq \xi_2 \leq x_2 \leq \cdots \leq x_{n-1} \leq \xi_n \leq x_n = b.$$

The pair $(\underline{x}, \underline{\xi})$ is referred to as a tagged partition.



Example 11
Consider the tagged partition $(\underline{x}, \underline{\xi}) = (\{1, 3/2, 2, 3\}, \{5/4, 7/4, 5/2\}$. Then,

$$\|\underline{x}\| = \max\{3/2 - 1, 2 - 3/2, 3 - 2\} = 1.$$

Definition 12 (Riemann sum)
The Riemann sum of f corresponding to $(\underline{x}, \underline{\xi})$ is the number

$$S_f(\underline{x}, \underline{\xi}) := \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}).$$

Let's try to understand how to interpret this using a picture. For $f \in C([a, b])$ positive, $S_f(\underline{x}, \underline{\xi})$ is an approximate area under the graph of f. As $\|\underline{x}\| \to 0$, we *should* expect these approximate areas to converge to a number A, which we **interpret** as the area under the curve f on the interval $[a, b]$.

Question 13. *Do these approximate sums actually converge?*

We will answer this question and more during the next few lectures.