18.100A: Complete Lecture Notes

Lecture 23:

Pointwise and Uniform Convergence of Sequences of Functions

Sequences of Function

Power Series

Remark 1. *Power series motivate the general discussion of sequences of functions.*

Definition 2 (Power series)

A power series about x_0 is a series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Theorem 3

Suppose

$$R = \lim_{m \to \infty} |a_m|^{1/n}$$

exists, and let

$$p = \begin{cases} \frac{1}{R} & R > 0 \\ \infty & R = 0. \end{cases}$$

Then, $\sum a_m (x - x_0)^m$ converges absolutely if $|x - x_0| < p$ and diverges if $|x - x_0| > p$.

Definition 4 (Radius of Convergence)

In the above theorem, we define p to be the radius of convergence.

Proof: We have

$$\lim_{n \to \infty} |a_m (x - x_0)^m|^{1/m} = R|x - x_0|,$$

and the theorem follows by the Root test. □

Suppose $\sum a_m (x - x_0)^m$ is a power series with radius of convergence p. Furthermore, define $f : (x_0 - p, x_0 + p) \to \mathbb{R}$ such that

$$f(x) := \sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Then, f is a limit of a sequence of functions

$$f(x) = \lim_{n \to \infty} f_n(x),$$

for $x \in (x_0 - p, x_0 + p)$ and where

$$f_n(x) = \sum_{m=0}^{n} a_m (x - x_0)^m.$$

Example 5

For example, we have

$$f(x) = \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m.$$

Question 6. *This concept begs a number of questions:*

1. *Is f continuous?*
2. *Is f differentiable, and does $f' = \lim_{n\to\infty} f_n'$?*
3. *If 1. is true, does*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n?$$

These questions will be the key motivator for the last section of this course.

Pointwise and Uniform Convergence

We now consider a setting far more general than power series.

Definition 7 (Pointwise Convergence)

For $n \in \mathbb{N}$, let $f_n : S \to \mathbb{R}$. Let $f : S \to \mathbb{R}$. We say that $\{f_n\}$ converges pointwise to f if for all $x \in S$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Let's consider some examples.

1. Let $f_n(x) = x^n$ on [0,1]. Then,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}.$$

   Thus, $\{f_n\}$ converges to the above pointwise function. Hence, notice that a sequence of continuous functions may not converge pointwise to a continuous function!

2. Let $f_n(x) = \sum_{m=0}^{n} x^m$ for $x \in (-1,1)$. Then,

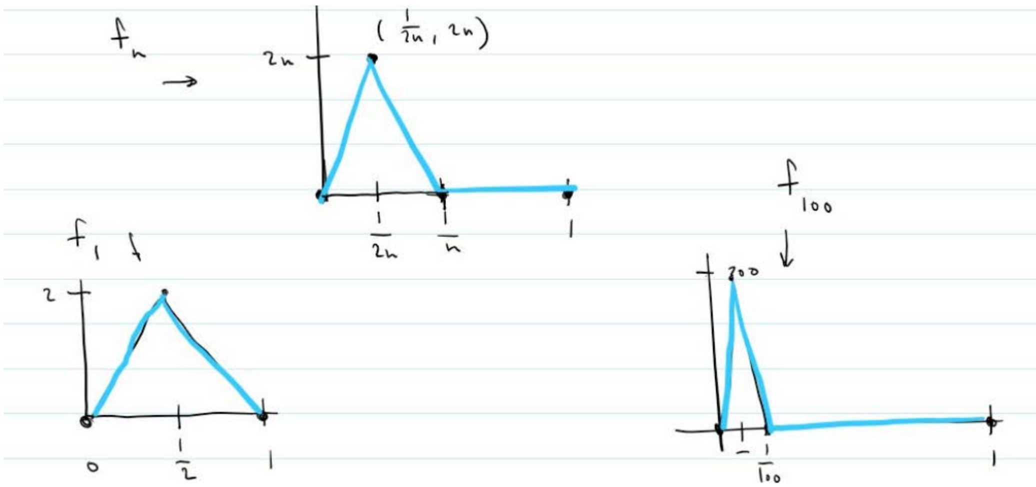
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sum_{m=0}^{n} x^m = \frac{1}{1-x}.$$

   Hence, pointwise, this sequence converges to its power series (see the above example).

3. Let $f_n(x) : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 4n^2 x & x \in \left[0, \frac{1}{2n}\right] \\ 4n - 4n^2 x & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & x \in \left[\frac{1}{n}, 1\right] \end{cases}.$$

   We can picture this sequence (on the next page)



Then, $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$. Let $x \in (0, 1]$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < x$. Then, for all $n \geq N$,

$$f_n(x) = 0.$$

Therefore,

$$\{f_n(x)\} = f_1(x), \ldots, f_{N-1}(x), 0, 0, 0, \ldots.$$

Hence, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$. Thus, $\{f_n\}$ converges pointwise to $f(x) = 0$ on $[0, 1]$.

Definition 8 (Uniform Convergence)

For $n \in \mathbb{N}$, let $f_n : S \to \mathbb{R}$, and let $f : S \to \mathbb{R}$. Then, we say f_n converges to f uniformly or **converges uniformly to** f if $\forall \epsilon > 0\ \exists M \in \mathbb{N}$ such that for all $n \geq M\ \forall x \in S$,

$$|f_n(x) - f(x)| < \epsilon$$

Theorem 9

If $f_n : S \to \mathbb{R}$, $f : S \to \mathbb{R}$, and $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

Proof: Let $c \in S$ and let $\epsilon > 0$. Then, $f_n \to f$ uniformly implies that there exists $M_0 \in \mathbb{N}$ such that for all $n \geq M, \forall x \in S$, $|f_n(x) - f(x)| < \epsilon$. Choose $M = M_0$. Then, $\forall n \geq M$,

$$|f_n(c) - f(c)| < \epsilon.$$

Thus, $\lim_{n\to\infty} f_n(c) = f(c)$ for all $c \in S$, and therefore $f_n \to f$ pointwise. □