18.100A: Complete Lecture Notes

Lecture 25:

Power Series and the Weierstrass Approximation Theorem

Last time, we asked three questions about interchanging limits:

Question 1. *Hence, we ask three questions about interchanging limits:*

1. *If $f_n : S \to \mathbb{R}$, f_n continuous and $f_n \to f$ pointwise or uniform, then is f continuous?*
2. *If $f_n : [a, b] \to \mathbb{R}$, f_n differentiable, and $f_n \to f$ with $f_n' \to g$, then is f differentiable and does $g = f'$?*
3. *If $f_n : [a, b] \to \mathbb{R}$, with f_n and f continuous such that $f_n \to f$, then does*

$$\int_a^b f_n = \int_a^b f?$$

The answer to the above questions are all **no**, if the convergence is pointwise as seen by the following counterexamples:

1. Let $f_n(x) = x^n$ on $[0, 1]$ is continuous $\forall n$. As we noted earlier, $f_n(x) \to f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$. Notice that f is not continuous.
2. Let $f_n(x) = \frac{x^{n+1}}{n+1}$ on $[0, 1]$. Then, $f_n \to 0$ pointwise on $[0, 1]$. However,

$$f_n'(x) \to g(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}.$$

   Thus, $g(x) \neq (0)' = 0$ at $x = 1$.

3. Consider the functions

$$f_n(x) = \begin{cases} 4n^2 x & x \in \left[0, \frac{1}{2n}\right] \\ 4n - 4n^2 x & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & x \in \left[\frac{1}{n}, 1\right] \end{cases}$$

   as described in the previous lecture. Then, $f_n(x) \to 0$ pointwise on [0,1] as we showed last time. However,

$$\int_0^1 f_n = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2n} \cdot 2n = 1 \not\to 0 = \int_0^1 0.$$

We now prove that the answer to the three questions above is **yes** if convergence is uniform.

Theorem 2

If $f_n : S \to \mathbb{R}$ is continuous for all n, $f : S \to \mathbb{R}$, and $f_n \to f$ uniformly, then f is continuous.

Proof: Let $c \in S$ and let $\epsilon > 0$. Since $f_n \to f$ uniformly, $\exists M \in \mathbb{N}$ such that $\forall n \geq M$, $\forall y \in S$,

$$|f_n(y) - f(y)| < \frac{\epsilon}{3}.$$

Since $f_M : S \to \mathbb{R}$ is continuous, $\exists \delta_0 > 0$ such that $\forall |x - c| < \delta_0$,

$$|f_M(x) - f_M(c)| < \frac{\epsilon}{3}.$$

Choose $\delta = \delta_0$. If $|x - c| < \delta$, then

$$\begin{aligned}|f(x) - f(c)| &\leq |f(x) + f_M(x)| + |f_M(c) - f(c)| + |f_M(x) - f_M(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

□

Theorem 3

If $f_n : [a, b] \to \mathbb{R}$ is continuous for all n, $f : [a, b] \to \mathbb{R}$ and $f_n \to f$ uniformly, then

$$\int_a^b f_n \to \int_a^b f.$$

Proof: Let $\epsilon > 0$. Since $f_n \to f$ uniform, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0, \forall x \in [a, b]$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{b - a}.$$

Then, for all $n \geq M = M_0$, we have

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \int_a^b |f_n - f| < \int_a^b \frac{\epsilon}{b - a} = \epsilon.$$

□

Remark 4. *Notationally, this states that*

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b \lim_{n \to \infty} f_n = \int_a^b f.$$

Theorem 5

If $f_n : [a, b] \to \mathbb{R}$ is continuously differentiable, $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$, and

$$\begin{aligned}f_n &\to f \quad \text{pointwise}, \\ f_n' &\to g \quad \text{uniformly},\end{aligned}$$

then f is continuously differentiable and $g = f'$.

Proof: By the FTC, $\forall n \forall x \in [a, b]$,

$$f_n(x) - f_n(a) = \int_a^x f_n'.$$

Thus, by the previous two theorems,

$$\begin{aligned}f(x) - f(a) &= \lim_{n \to \infty} (f_n(x) - f_n(a)) \\ &= \lim_{n \to \infty} \int_a^x f_n' \\ &= \int_a^x g.\end{aligned}$$

Therefore, $f(x) = f(a) + \int_a^x g$. Thus, by the FTC, f is differentiable and $f' = (\int_a^x g)' = g$. □

We now return back to our study of power series, answering some questions we asked at the beginning of Lecture 23.

Theorem 6

Let $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ be a power series of radius of convergence $p \in (0, \infty]$. Then, $\forall r \in (0, p)$, $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ converges uniformly on $[x_0 - r, x_0 + r]$.

Proof: Let $r \in [0, p)$. Then, $\forall j \in \mathbb{N} \cup \{0\}$, $\forall x \in [x_0 - r, x_0 + r]$,

$$|a_j(x - x_0)^j| \leq |a_j| r^j =: M_j.$$

Now,

$$\lim j \to \infty M_j^{1/j} = \lim_{j\to\infty} |a_j|^{1/j} r = \begin{cases} \frac{r}{p} & p < \infty \\ 0 & p = \infty \end{cases}$$

since $p^{-1} = \lim_{j\to\infty} |a_j|^{1/j}$. Since $r < p$, we have

$$\lim_{j\to\infty} M_j^{1/j} < 1 \implies \sum_{j=0}^{\infty} M_j \text{ converges.}$$

By the Weierstrass M-test, it follows that $\sum_{j=0}^{\infty} a_j(x - x_0)^j$ converges uniformly on $[x_0 - r, x_0 + r]$. □

Theorem 7

Let $\sum_{j=0}^{\infty} a_j(x - x_0)^j$ be a power series with radius of convergence $p \in (0, \infty]$. Then,

1. $\forall c \in (x_0 - p, x_0 + p)$, $\sum_{j=0}^{\infty} a_j(x - x_0)^j$ is differentiable at c and

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=0}^{\infty} a_j(x - x_0)^j = \sum_{j=0}^{\infty} j a_j (x - x_0)^{j-1}.$$

2. $\forall a, b$ such that $x_0 - p < a < b < x_0 + p$,

$$\int_a^b \sum_{j=0}^{\infty} a_J(x - x_0)^j \,\mathrm{d}x = \sum_{j=0}^{\infty} a_j \left(\frac{(b - x_0)^{j+1}}{j+1} - \frac{(a - x_0)^{j+1}}{j+1} \right).$$

Remark 8. *Since*

$$\lim_{j\to\infty} ((j+1)|a_{j+1}|)^{1/j} = \lim_{j\to\infty} \left(((j+1)|a_{j+1}|^{1/(j+1)} \right)^{(j+1)/j} = \lim_{k\to\infty} |a_k|^{1/k} = p,$$

1. implies $\sum a_j(x - x_0)^j$ *is infinitely differentiable and*

$$k! a_k = \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} \sum a_j(x - x_0)^j \right) \Big|_{x=x_0}.$$

Weierstrass Approximation Theorem

Remark 9. *This theorem essentially states: "Every continuous function on* $[a, b]$ *is almost a polynomial."*

Theorem 10 (Weierstrass Approximation Theorem)
If $f \in C([a,b])$, there exists a sequence of polynomials $\{P_n\}$ such that

$$P_n \to f \quad \text{uniformly on} \quad [a,b].$$

The idea of the proof is to choose a suitable sequence of polynomials $\{Q_n\}_n$ such that Q_n behaves like a 'Dirac delta function' as $nto\infty$. Then, the sequence of polynomials $P_n(x) = \int_0^1 Q_n(x-t)f(t)\,\mathrm{d}t$ converges to $f(x)$ as $n \to \infty$. We will prove this momentarily, but first we need to do the ground work.

Notice that we only need to consider $a = 0$ and $b = 1$, with $f(0) = f(1) = 0$. If we prove this case, then for a general $\tilde{f} \in C([0,1])$, $\exists$ a sequence of polynomials

$$P_n(x) \to \tilde{f}(x) - \tilde{f}(0) - x(\tilde{f}(1) - \tilde{f}(0)) \quad \text{uniformly.}$$

Hence,

$$\tilde{P}_n(x) = P_n(x) + \tilde{f}(0) + x(\tilde{f}(1) - \tilde{f}(0)) \to \tilde{f}(x) \quad \text{uniformly.}$$

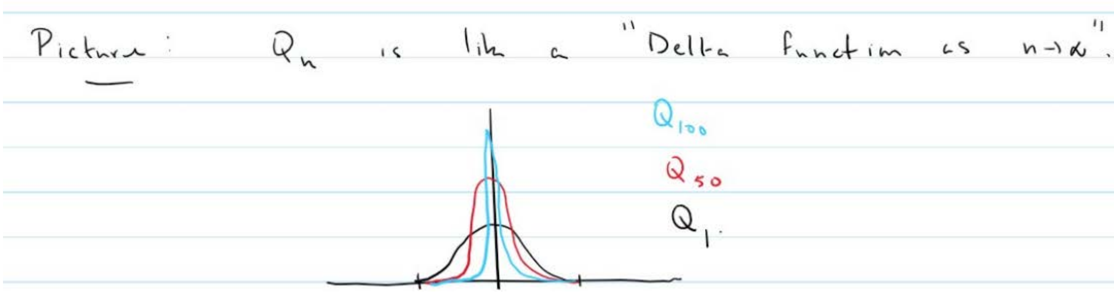
Theorem 11
Let $c_n := (\int_{-1}^1 (1-x^2)^n \,\mathrm{d}x)^{-1} > 0$, and let

$$Q_n(x) = c_n(1-x^2)^n.$$

Then,

1. $\forall n$, $\int_{-1}^1 Q_n = 1$.
2. $\forall n$, $Q_n(x) \geq 0$ on $[-1,1]$, and
3. $\forall \delta \in (0,1)$, $Q_n \to 0$ uniformly on $\delta \leq |x| \leq 1$.

Before we prove this, here is a picture of Q_n:



Proof:

2. Immediately clear.
1. $\int_{-1}^1 Q_n = c_n \int_{-1}^1 (1-x^2)^n \,\mathrm{d}x = 1$ by definition of c_n.
3. We first estimate c_n. We have for all $n \in \mathbb{N}$ and $\forall x \in [-1,1]$,

$$(1-x^2)^n \geq 1 - nx^2.$$

We proved this way earlier in the course by induction, but it also follows from the calculus we have proven as

$$g(x) = (1-x^2)^n - (1-nx^2)$$

satisfies $g(0) = 0$, and

$$g'(x) = n \cdot 2x(1 - (1 - x^2)^{n-1}) \geq 0$$

in [0,1]. Thus, $g(x) \geq 0$ by the MVT.

Then,

$$\begin{aligned}
\frac{1}{c_n} &= \int_{-1}^{1} (1 - x^2)^n \, \mathrm{d}x \\
&= 2 \int_0^1 (1 - x^2)^n \, \mathrm{d}x \\
&> 2 \int_0^{1/\sqrt{n}} (1 - x^2)^n \, \mathrm{d}x \\
&\geq 2 \int_0^{1/\sqrt{n}} (1 - nx^2) \, \mathrm{d}x \\
&= 2 \left(\frac{1}{\sqrt{n}} - \frac{n}{3} \cdot n^{-3/2} \right) \\
&= \frac{4}{3}\sqrt{n} > \sqrt{n}.
\end{aligned}$$

Therefore, $c_n < \sqrt{n}$.

Let $\delta > 0$. We note $\lim_{n\to\infty} \sqrt{n}(1 - \delta^2)^n = 0$. Then,

$$\begin{aligned}
\lim_{n\to\infty} (\sqrt{n}(1 - \delta^2)^n)^{1/n} &= \lim_{n\to\infty} (n^{1/n})^{1/2}(1 - \delta^2) \\
&= 1 - \delta^2 < 1.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \sqrt{n}(1 - \delta^2)^n = 0.$$

Let $\epsilon > 0$, and choose $M \in \mathbb{N}$ such that for all $n \geq M$,

$$\sqrt{n}(1 - \delta^2)^n < \epsilon.$$

Then, $\forall n \geq M$ and $\forall \delta \leq |x| \leq 1$,

$$|c_n(1 - x^2)^n| < \sqrt{n}(1 - x^2)^n \leq \sqrt{n}(1 - \delta^2)^n < \epsilon.$$

□

We now prove the Weierstrass Approximation Theorem.

Proof: Suppose $f \in C([0, 1])$, $f(0) = f(1) = 0$. We extend f to an element of $C(\mathbb{R})$ by setting $f(x) = 0$ for all $x \notin [0, 1]$. We furthermore define

$$\begin{aligned}
P_n(x) &= \int_0^1 f(t) Q_n(t - x) \, \mathrm{d}t \\
&= \int_0^1 f(t) c_n (1 - (t - x)^2)^n \, \mathrm{d}t.
\end{aligned}$$

Note that $P_n(x)$ is in fact a polynomial.

Furthermore, observe that for $x \in [0,1]$,

$$\begin{aligned} P_n(x) &= \int_0^1 f(t)Q_n(t-x)\,\mathrm{d}t \\ &= \int_{-x}^{1-x} f(x+t)Q_n(t)\,\mathrm{d}t \\ &= \int_{-1}^{1} f(x+t)Q_n(t)\,\mathrm{d}t. \end{aligned}$$

The second equality is true by a change of variable, and the last equality is true as $f(x+t) = 0$ for $t \notin [-x, 1-x]$.

We now prove $P_n \to f$ uniformly on $[0,1]$. Let $\epsilon > 0$. Since f is uniformly continuous on $[0,1]$, $\exists \delta > 0$ such that $\forall |x-y| \leq \delta$, $|f(x) - f(y)| < \frac{\epsilon}{2}$. Let $C = \sup\{f(x) \mid x \in [0,1]\}$, which exists by the Min/Max theorem i.e. the EVT. Choose $M \in \mathbb{N}$ such that $\forall n \geq M$,

$$\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{8C}.$$

Thus, $\forall n \geq M, \forall x \in [0,1]$, by the previous theorem,

$$\begin{aligned} |P_n(x) - f(x)| &= \left| \int_{-1}^{1} (f(x-t) - f(t))Q_n(t)\,\mathrm{d}t \right| \\ &\leq \int_{-1}^{1} |f(x-t) - f(x)|Q_n(t)\,\mathrm{d}t \\ &\leq \int_{|t|\leq\delta} |f(x-t) - f(x)|Q_n(t)\,\mathrm{d}t + \int_{\delta\leq|t|\leq 1} |f(x-t) - f(x)|Q_n(t)\,\mathrm{d}t \\ &\leq \frac{\epsilon}{2}\int_{|t|\leq\delta} Q_n(t)\,\mathrm{d}t + \sqrt{n}(1-\delta^2)^n \int_{\|\delta|\leq|t|\leq 1} 2C \\ &< \frac{\epsilon}{2} + 4C\sqrt{n}(1-\delta^2)^n \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

In the last minute of the course, Casey Rodriguez stated: "This was quite an experience; teaching to an empty room. I hope you did get something out of this class. Unfortunately I wasn't able to meet a lot of you, and that's one of the best parts of teaching...."