Real Analysis
Midterm

October 16, 2020

The following exam consists of **5 problems** worth **15 points each**. Solutions should be written in complete sentences where appropriate.

The midterm is open book, open notes, but **collaborating with other students or the internet is strictly prohibited**. By signing your name below, you attest to following these rules for the exam. Evidence to the contrary will be treated as academic misconduct and will responded to according to MIT Institute Policy 10.2.

Name:

1. (a) (5 points) Let $f : A \to B$, and let C, D be subsets of B. Prove

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

(b) (5 points) When E is a countable subset of $\mathbb{R}$, is the complement $\mathbb{R} \setminus E$ always uncountable? Explain why or why not.

(c) (5 points) When E is an uncountable subset of $\mathbb{R}$, is the complement $\mathbb{R} \setminus E$ always countable? Explain why or why not.

2. A subset of real numbers $U \subset \mathbb{R}$ is *open* if for all $x \in U$, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. A subset of real numbers $F \subset \mathbb{R}$ is *closed* if F^c is open.

   (a) (5 points) State what it means to say U is *not open*.

   (b) (5 points) Prove that if U is not open, then there exists $x \in U$ and a sequence $\{x_n\}_n$ of elements of U^c such that

$$\lim_{n \to \infty} x_n = x.$$

(c) (5 points) Suppose $F \subset \mathbb{R}$ has the following property: for every convergent sequence $\{x_n\}_n$ of elements of F we have $\lim_{n\to\infty} x_n \in F$. Prove that F is closed.
Hint: Argue by contradiction using (b).

3. (a) (5 points) Use the definition of convergence to prove that

$$\lim_{n\to\infty} \frac{10n^2}{n^2+16n+1} = 10.$$

(b) For each of the following scenarios, give an example satisfying the stated property. Formal proofs are not required, but some explanation may be useful.

(i) (5 points) A sequence $\{x_n\}$ converging to 0 which is not monotonic.

(ii) (5 points) An unbounded sequence that has a convergent subsequence.

4. (a) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences of real numbers.

   (i) (5 points) Prove that the sequence $\{x_n + y_n\}$ is bounded.

   (ii) (5 points) Prove that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(b) (5 points) Let E denote the set of all real numbers in $(0,1)$ with decimal expansion involving only 1's and 2's:

$$E = \{x \in (0,1) : \forall j \in \mathbb{N}, \exists d_{-j} \in \{1,2\}, \text{ such that } x = 0.d_{-1}d_{-2}\ldots\}.$$

Note that $0.2 \notin E$ but $0.222222\ldots \in E$. Prove that $0.1111111\ldots$ is a cluster point of E.

5. (a) (5 points) Suppose that for all $n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0.$$

Prove that $\sum a_n$ converges if and only if $\sum b_n$ converges.

(b) Find all real numbers x such that the series converges. Find all real numbers x such that the series converges absolutely.

(i)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2020n}(x-10)^n$$

(ii)

$$\sum_{n=0}^{\infty} n! x^{n!}$$