Recitation 02

To-do list:

1. Discuss idea behind supremum and infimum proofs, with question 7 of the homework as an example.

2. Discuss diagonalization arguments.

Let's start, where else, but the beginning. With infimum and supremum proofs, we are often asked to show that the supremum and/or the infimum exists and then show that they satisfy a certain property. We had a similar problem during the first recitation:

> **Problem 1**
> Given $A, B \subset \mathbb{R}_{>0}$ are bounded and nonempty sets, and given the set $C := \{a + b \mid a \in A,\ b \in B\}$, show that $\sup C = \sup A \cdot \sup B$.

However, we did this problem last time, and will focus on a new one: Problem 7 from PSET 2.

> **Problem 2**
> Show that the set
> $$X := \{a \in \mathbb{R} \mid a^3 < 2, a > 0\}$$
> has a supremum, and that $(\sup X)^3 = 2$.

In these sorts of proofs, there are few key steps:

1. Show that the set X is bounded, which them implies it must have a supremum as a property of the real numbers.

2. Show that $\sup X = r \in \mathbb{R}$. To do this, we usually show that $\sup X \geq r$ and $\sup X \leq r$.

Let's go through these steps in this problem. Based on the very definition of the set X, we know it must be bounded, as there doesn't exist an element of X less than 0, and there certainly aren't any elements greater than 3 (the choice of 3 is arbitrary but sufficient to show it is bounded). Thus, we just have to do step 2. Here, we want to show that $(\sup X)^3 = 2$, and thus want to show that $(\sup X)^3 \leq 2$ and $(\sup X)^3 \geq 2$. Let $\sup X = r$ for simplicity.

Assume for the sake of contradiction that $r^3 > 2$. Let's try to intuitively understand why this should give a contradiction. Given this is the case, then we should be able to subtract a "small" $h > 0$ from r such that $(r - h)^3 > 2$, and then $r - h$ is still an upper bound for X. This would be a contradiction, as the supremum is the **least** upper bound. However, we still need to make this proof more formal by explicitly saying what h is. To do this, consider the following:

$$\begin{aligned}(r - h)^3 &= r^3 - 3r^2h + 3rh^2 - h^3 \\ &= r^3 - 3r^2h + h^2(3r - h).\end{aligned}$$

Notice that if we can find an h such that $r^3 - 3r^2h \geq 2$, and such that $3r - h > 0$, then we will have that $(r - h)^3 > 2$. This would be the contradiction we discussed earlier, and then $(\sup X)^3 \leq 2$. We would have to do something similar for $r^3 < 2$ and reach a contradiction, but at this point we leave the rest as an exercise for the reader.

Now, lets discuss the second item on our to-do list: Diagonalization Arguments. We can use this sort of argument to prove some really statements, such as the following:

A For any set A, $|\mathcal{P}(A)| > |A|$.

B The set of real numbers is uncountable, i.e. $|\mathbb{R}| > |\mathbb{N}|$.

We will first prove statement B, and then statement A.

Assume that there exists a bijection between $\mathbb{R}$ and $\mathbb{N}$. If this is the case, then we can list the elements of $\mathbb{R}$ and put them into a table:

$\mathbb{N}$	$\mathbb{R}$					
1	**2.**	7	1	8	2	...
2	1.	**6**	1	8	0	...
3	3.	1	**4**	1	5	...
⋮			⋮			

For each real number in the row i, take the i-th digit (as shown by the bold numbers in the table above). Then consider the real number with those diagonal elements: $x = 2.64\ldots$. x differs from the i-th real number in the i-th decimal point, which implies that x cannot be on this table! This is a contradiction, as we assumed that every real number was on this list (as we assumed there existed a bijection). Therefore, $\mathbb{R}$ is uncountable and furthermore $|\mathbb{R}| > |\mathbb{N}|$. □

Now we prove statement A in a similar way. Assume that there exists a bijection

$$g : A \to \mathcal{P}(A).$$

Consider the set $T = \{a \in A \mid a \notin g(a)\} \in \mathcal{P}(A)$. For every $a \in A$, either a is in T or not. If a is in T, then a is not in $g(a)$, and then $T \neq g(a)$. On the other hand if a is not in T, then a is in $g(a)$, which again means $T \neq f(a)$. In any case, $T \neq g(a)$. Therefore, there exists an element of $\mathcal{P}(A)$ that isn't achieved by the bijection g, which means g is not a bijection. One can see this argument through a table just as we did to show $|\mathbb{R}| > |\mathbb{N}|$. □ Here, T is

$\mathcal{P}(A)$ A	x_1	x_2	x_3	...
$g(x_1)$	**1**	1	0	...
$g(x_2)$	0	**0**	1	...
$g(x_3)$	1	0	**1**	...
⋮		⋮		

the set x_is that give value 1 on the diagonal (again shown with the bold numbers). □

18.100A / 18.1001 Real Analysis
Fall 2020