Recitation 05

To-do list:

1. First we will answer any questions there may be on PSETs and lectures.

2. Discuss $\epsilon - \delta$ limits with some examples.

We first discussed PSET 3 Problem 5: proving whether or not $\mathbb{Q}$ is open. Recall that a set $U \subset \mathbb{R}$ is open if $\forall x \in U$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. In this problem, we have $U = \mathbb{Q}$. Well, if $\mathbb{Q}$ is open, then for all $q \in \mathbb{Q}$, then there exist an $\epsilon > 0$ such that $(q - \epsilon, q + \epsilon) \subset \mathbb{Q}$. Is this true? It isn't! Consider problem 1, in which we saw that for all $x, y \in \mathbb{R}$ with $x < y$, there exists an $r \in \mathbb{R} \setminus \mathbb{Q}$ with $x < r < y$. Given $q - \epsilon, q + \epsilon \in \mathbb{R}$ for all $\epsilon > 0$, there must exist an $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $q - \epsilon < r < q + \epsilon$, and hence $r \in (q - \epsilon, q + \epsilon)$. Therefore, for all $\epsilon > 0$, $(q - \epsilon, q + \epsilon) \not\subset \mathbb{Q}$.

We now move on to the second item on our agenda: $\epsilon - \delta$ limits. We actually did an $\epsilon - \delta$ proof in the last recitation (simply chance ϵ' to δ in that proof. Let's try and build up the intuition behind these sorts of problems.

Example 3

Suppose that $x_n > 0$ for all $n \in \mathbb{N}$ and suppose that $x_n \to x > 0$. Show that

$$x_n^{\frac{1}{3}} \to x^{\frac{1}{3}}.$$

To rephrase this, we want to show that for all $\epsilon > 0$, there exists N such that $\forall n > N$, $|x_n^{\frac{1}{3}} - x^{\frac{1}{3}}| < \epsilon$ $(*)$. All that we know is that for all $\delta > 0$, $\exists N'$ such that $\forall n > N'$, $|x_n - x| < \delta$ (as $x_n \to x$).

Let's rewrite $(*)$. Using the fact that

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$

we can state that

$$|x_n^{\frac{1}{3}} - x^{\frac{1}{3}}| = \frac{|x_n - x|}{|x_n^{\frac{2}{3}} + x_n^{\frac{1}{3}} \cdot x^{\frac{1}{3}} + x^{\frac{2}{3}}|}$$

(by letting $a = x_n^{\frac{1}{3}}$ and $b = x^{\frac{1}{3}}$). Note that the denominator is nonzero based off of assumptions in the problem. (Note that it is usually a good sign when our approach utilizes all/most of the constraints on a problem.) Furthermore, by making the denominator smaller we make the fraction bigger, and hence we have

$$|x_n^{\frac{1}{3}} - x^{\frac{1}{3}}| = \frac{|x_n - x|}{|x_n^{\frac{2}{3}} + x_n^{\frac{1}{3}} \cdot x^{\frac{1}{3}} + x^{\frac{2}{3}}|} \leq \frac{|x_n - x|}{x^{\frac{2}{3}}}.$$

Using the fact that $x_n \to x$, we can pick $\delta = \epsilon \cdot x^{\frac{2}{3}} > 0$. Hence, given this value of δ, $\exists N'$ such that

$$|x_n - x| < \delta = \epsilon \cdot x^{\frac{2}{3}}.$$

Hence, for all $\epsilon > 0$ and $n > N'$,

$$|x_n^{\frac{1}{3}} - x^{\frac{1}{3}}| = \frac{|x_n - x|}{|x_n^{\frac{2}{3}} + x_n^{\frac{1}{3}} \cdot x^{\frac{1}{3}} + x^{\frac{2}{3}}|} \leq \frac{|x_n - x|}{x^{\frac{2}{3}}} \leq \frac{\epsilon \cdot x^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \epsilon.$$

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