Final Exam Review

To-do list:

1. First we will cover continuity and uniform continuity,
2. Then we will derivatives, and how to use the definition of a derivative to calculate it.
3. Finally we will discuss pointwise and uniform convergence, and how to prove these.

Note that these aren't all of the topics that may be on the exam. The following will not be covered during this recitation, but may show up:

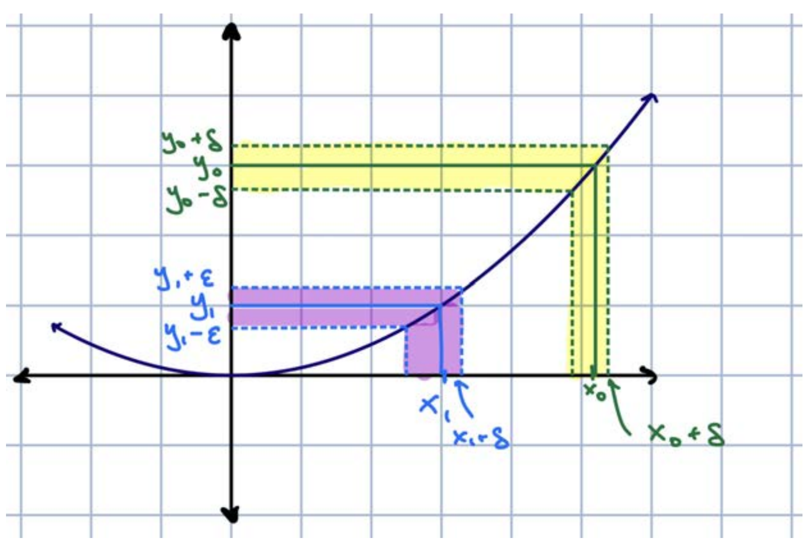
- Materials from the first half of the class (covered during the Midterm Exam Review).
- The Mean Value Theorem, the Fundamental Theorem of Calculus, and the Intermediate Value Theorem
- L'Hospital's Rule.

Continuity and Uniform Continuity

Lets start with the definitions. Let $f : S \to \mathbb{R}$. Then,

- f is continuous at the point $x_0 \in S$ if $\forall \epsilon > 0$, $\exists \delta$ such that $|f(x) - f(x_0)| < \epsilon$ for $|x - x_0| < \delta$.
- f is a continuous function if it is continuous for every $x \in S$. This is the definition of **pointwise continuity**. Note that with pointwise continuity, δ *can depend* on x_0.
- f is uniformly continuous on S if $\forall \epsilon > 0$ there exists $\delta > 0$ such that $\forall x_0 \in S$, $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$. The difference between this and the definition of pointwise continuity is where the x_0 is in the mathematical statement– since the "$\forall x_0$" occurs after the "δ", this means δ **cannot** depend on x_0.

An example of a non-uniformly continuous function is $f(x) = x^2$, as given $\epsilon > 0$, δ depends on x. We can see this pictorally, but will not prove it rigorously here.



Problem 10

Show that the function $f(x) = \frac{1}{x^2+1}$ is a uniformly continuous function on $\mathbb{R}$ (i.e. $S = \mathbb{R}$).

Proof: We want to show that $\forall \epsilon > 0$ there exists $\delta > 0$ such that $\forall x_0 \in S$, $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$. Thus, consider the following:

$$\begin{aligned}\frac{1}{x^2+1} - \frac{1}{x_0^2+1} &= \frac{|x_0^2 + 1 - x^2 - 1}{|x^2+1| \cdot |x_0^2+1|} \\ &= \frac{|x_0 - x| \cdot |x_0 + x|}{(x^2+1)(x_0^2+1)}.\end{aligned}$$

Now we have an $|x - x_0|$ in the numerator, which is useful for us to use δs and ϵs for. Now note that

$$\frac{|x_0 + x|}{(x^2+1)(x_0^2+1)} \to 0$$

as $x \to \pm\infty$. Furthermore, the function is continuous as a function of x and is thus bounded. Hence, $\exists B$ such that

$$\frac{|x + x_0|}{(x^2+1)(x_0^2+1)} < B$$

for all x and x_0. Thus, let $\delta = \frac{\epsilon}{B}$ (notably, δ is independent of x_0). Therefore, $\forall \epsilon > 0$,

$$\begin{aligned}\frac{1}{x^2+1} - \frac{1}{x_0^2+1} &= \frac{|x_0^2 + 1 - x^2 - 1}{|x^2+1| \cdot |x_0^2+1|} \\ &= \frac{|x_0 - x| \cdot |x_0 + x|}{(x^2+1)(x_0^2+1)} \\ &\leq B|x_0 - x| \\ &< B\delta \\ &= \epsilon.\end{aligned}$$

Hence, f is uniformly continuous. $\square$

Derivatives

We will first use an example to show how we can calculate a derivative strictly using the definition.

Problem 11

Let

$$f(x) = \begin{cases} \sqrt{x^2+1}, & x \geq 0 \\ x^2 + 1, & x < 0 \end{cases}.$$

Compute f', and prove whether or not f'' is defined on all of $\mathbb{R}$.

Proof: To calculate f', we first calculate f' for $x > 0$ and $x < 0$ using standard calculus: For $x > 0$

$$\begin{aligned}f'(x) &= \frac{2x}{2\sqrt{x^2+1}} \\ &= \frac{x}{\sqrt{x^2+1}}.\end{aligned}$$

For $x < 0$,

$$f'(x) = 2x.$$

This is not meant to be the difficult analysis part of the problem– what makes it more tricky is finding the derivative at $x_0 = x$. To do this, we want to find

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 1}{x}.$$

To do so, we find the left and right sided limits and show they are equal.

$$\begin{aligned}
\lim_{x \to 0^+} \frac{f(x) - 1}{x} &= \lim_{x \to 0^+} \frac{\sqrt{x^2+1} - 1}{x} \\
&= \lim_{x \to 0^+} \frac{\sqrt{x^2+1} - 1}{x} \cdot \frac{\sqrt{x^2+1} + 1}{\sqrt{x^2+1} + 1} \\
&= \lim_{x \to 0^+} \frac{x^2}{x \cdot \sqrt{x^2+1} + 1} \\
&= 0. \\
\lim_{x \to 0^-} \frac{f(x) - 1}{x} &= \lim_{x \to 0^+} \frac{x^2 + 1 - 1}{x} \\
&= 0.
\end{aligned}$$

Therefore, f is differentiable at 0 and $f'(0) = 0$. Thus we can say

$$f'(x) = \begin{cases} \frac{x}{\sqrt{x^2+1}}, & x \geq 0 \\ 2x, & x < 0 \end{cases}.$$

Now we want to see whether or not f'' is defined on all of $\mathbb{R}$, and to start we once again calculate f'' for $x > 0$, $x < 0$, and see what happens at 0. For $x > 0$, by the quotient rule,

$$\begin{aligned}
f''(x) &= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} \\
&= \frac{x^2 + 1 - x^2}{(x^2+1)^{\frac{3}{2}}} \\
&= \frac{1}{(x^2+1)^{\frac{3}{2}}}.
\end{aligned}$$

For $x < 0$,

$$f''(x) = 2.$$

However, note that

$$1 = \lim_{x \to 0^+} f''(x) \neq \lim_{x \to 0^-} f''(x) = 2,$$

and thus $f''(x)$ is not defined at $x = 0$. $\square$

Problem 12

Calculate

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^{x^2} \sin(t^2) \, \mathrm{d}t.$$

Proof: Given the structure of this problem, it should be clear that we want to use the Fundamental Theorem of Calculus, but we have to be careful with that x^2. Firstly, we will restate the Fundamental Theorem. Let $F'(x) = f(x)$. Then,

$$\int_0^x f(t) \, \mathrm{d}t = F(x) - F(0),$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}(F(x) - F(0)) = f(x).$$

It will be helpful in this problem (and in general on exams!) to go a bit slowly. Let $F'(x) = \sin(x^2)$. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \sin(t^2)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}(F(x^2) - F(0)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}F(x^2).\end{aligned}$$

Using the chain rule and the fact that $F'(x) = \sin(x^2)$, we have

$$\begin{aligned}&= 2x \cdot F'(x^2) \\ &= 2x\sin(x^4).\end{aligned}$$

□

Pointwise vs Uniform Convergence

Firstly, we define both. Let $f_n : S \to \mathbb{R}$ be a sequence of functions.

- We say that f_n converges to f pointwise if for all $x \in S$,

  $$\lim_{n\to\infty} f_n(x) = f(x).$$

- We say that $f_n \to f$ uniformly if $\forall \epsilon > 0$, $\exists N$ such that

  $$\sup_x |f_n(x) - f(x)| < \epsilon$$

  for all $n > N$. Notice that this is true if and only if

  $$|f_n(x) - f(x)| < \epsilon$$

  for all $n > N$ and $\forall x \in S$. This statement may feel trivial, but allows us another way to prove uniform convergence.

Lets discuss both separately.

Example 13

Consider the sequence of functions $f_n(x) = x^n$ for $x \in [0, 1]$. Show that $f_n \to f$ pointwise where

$$f(x)\begin{cases} 0, & x \neq 1 \\ 1, & x = 1 \end{cases}.$$

Proof: Pick $x_0 \in [0, 1]$ and let $a_n = f_n(x_0) = x_0^n$. Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_0^n = \begin{cases} 0, & x \neq 1 \\ 1, & x = 1 \end{cases}.$$

□

Example 14

Consider the sequence of functions $f_n(x) = \frac{x}{n}$ for $x \in \mathbb{R}$. Show that $f_n \to f$ pointwise $f(x) = 0$ for all $x \in \mathbb{R}$.

Proof: Pick $x_0 \in \mathbb{R}$ and let $a_n = f_n(x_0) = \frac{x_0}{n}$. Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x_0}{n} = 0.$$

$\square$

Example 15

Consider the sequence of functions $f_n(x) = \frac{nx+1}{n(x^2+1)}$ for $x \in \mathbb{R}$. Show that $f_n \to f$ uniformly where

$$f(x) = \frac{x}{x^2+1}.$$

Proof: Well, firstly note that

$$f_n(x) = \frac{x}{x^2+1} + \frac{1}{n(x^2+1)} = f(x) + \frac{1}{n(x^2+1)}.$$

Furthermore, we want to show that for all $\epsilon > 0$, there exists an N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and $x \in \mathbb{R}$. Let N be the smallest natural number such that $\frac{1}{N} < \epsilon$ (we can similar have let $N = \lceil \frac{1}{\epsilon} \rceil$). Then,

$$|f_n(x) - f(x)| = \frac{1}{n(x^2+1)} \leq \frac{1}{n} < \frac{1}{N} < \epsilon$$

for all $n > N$. $\square$

To finish off this review, lets consider the similarities and differences of uniform continuity and convergence.

Uniform Continuity	Uniform Convergence
- Deals with a single function	- Deals with a sequence of functions
- It is continuous and "the same δ" works for all points	- "The same N" works for every point
	- These functions have a limit (namely another function)

Problem 16

Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function, and let

$$f_n(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,\mathrm{d}t.$$

Show that $f_n \to f$ uniformly.

Proof: One useful way to start approaching problems like this is to figure out where certain terms/coefficients come from. For instance, why do we have a $\frac{n}{2}$? Well, we can notice that

$$\frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,\mathrm{d}t = \frac{\int_0^{1+\frac{1}{n}} f(t)\,\mathrm{d}t - \int^{x-\frac{1}{n}} f(t)\,\mathrm{d}t}{x + \frac{1}{n} - (x - \frac{1}{n})}.$$

Let $F'(x) = f(x)$, and thus by the Fundamental Theorem of Calculus we have that

$$\begin{aligned}\frac{n}{2}\int_{x-\frac{1}{n}}^{1+\frac{1}{n}} f(t)\,\mathrm{d}t &= \frac{\int_0^{x+\frac{1}{n}} f(t)\,\mathrm{d}t - \int_0^{x-\frac{1}{n}} f(t)\,\mathrm{d}t}{x+\frac{1}{n}-(x-\frac{1}{n})} \\ &= \frac{F(x+\frac{1}{n}) - F(0) - (F(x-\frac{1}{n}) - F(0))}{x+\frac{1}{n}-(x-\frac{1}{n})} \\ &= \frac{F(x+\frac{1}{n}) - F(x-\frac{1}{n})}{x+\frac{1}{n}-(x-\frac{1}{n})}.\end{aligned}$$

Now we want to relate this back to f, and we can notice that what we have so far is in the form of the mean value theorem, which states that $\exists x_0 \in (x-\frac{1}{n}, x+\frac{1}{n})$ such that

$$\frac{F(x+\frac{1}{n}) - F(x-\frac{1}{n})}{x+\frac{1}{n}-(x-\frac{1}{n})} = F'(x_0) = f(x_0).$$

Therefore, for some $x_0 \in (x-\frac{1}{n}, x+\frac{1}{n})$, $f_n(x) = f(x_0)$. Hence, $|f_n(x) - f(x)| = |f(x_0) - f(x)|$. Since f is uniformly continuous, for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ when $|x - x_0| < \delta$. Let $\delta = \frac{1}{N}$.

Therefore, $f_n \to f$ uniformly. $\square$

Best of luck on your exam!

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