Midterm Exam Review

To-do list:

1. First we will cover general exam review tips.
2. Then we will discuss definitions, theorems, and problems, including
   - Suprema/infina
   - Limits
   - $\liminf/\limsup$ and Cauchy sequences
   - Series and series tests.

The first exam tip is to look back at definitions we have covered so far (there are a lot!). Try to interpret them intuitively, and if possible try to draw pictures. Think about why these definitions are useful, and in particular see how they were useful in homework problems.

Now we move onto some definitions and remarks about them.

Suprema and Infima

p is the **supremum** of a set S if it is the *least upper bound*. Recall that p is an upper bound for S if $p \geq x$ for all $x \in S$. If p is the supremum of S and p' is an upper bound for S, then $p \leq p'$. There are a few very properties and remarks we would like to cover here regarding suprema:

- p is the supremum of a set S if $\forall \epsilon > 0$ there exists an $x \in S$ such that $p - \epsilon < x \leq p$. (How can you prove this/think about this intuitively?)
- Finite sets always contain their supremum, and in fact $\sup S = \max S$ where S is a finite set. One example of an infinite set that does not contain the supremum is $(0, 1)$, which has supremum 1 but has **no** maximum.

There are a few algebraic operations we can consider:

- $\sup(\lambda S) = \lambda \sup S$ if $\lambda \geq 0$.
- $\sup(A + B) = \sup A + \sup B$ if $\sup A$ and $\sup B$ exist. This was a problem in the homework.
- If $A, B \subset R_{>0}$ then $\sup(AB) = \sup A \cdot \sup B$ (which we covered in Recitation 01).

Problem 6

Let X be a closed and bounded set. Show that X contains its supremum.

Try to solve this problem before you read the solution!

Proof:

We are given that X is closed and bounded, but what does this actually tell us?

1. X is closed $\iff$ X^c is open $\iff$ $\forall x \in X^c$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset X^c$.
2. X is bounded, which implies that X has a supremum. Let $\sup X = p$.

Most of the time, simply writing out/knowing these definitions is half of the battle. Now we will show that $p \in X$ by contradiction.

Suppose that $p \notin X$. Then, $p \in X^c$, which we know to be open by 1.. Therefore, there exists an $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset X^c$. This implies that there does not exist any $x \in X$ such that $p - \epsilon < x < p$. Therefore,

$p-\epsilon$ is an upper bound for X. However, $p-\epsilon<p \implies p \neq \sup X$ then p is not a least upper bound. This is a contradiction, and thus $p \in X$. $\square$

One more remark regarding lim sup. Firstly, the definition: $\limsup x_n = \lim \sup_{m \geq n}\{x_m\}$. Now: is it the case that the lim sup is always in the sequence? No. Consider the sequence $x_n = 1-\frac{1}{n}$. Then, $\limsup x_n = 1 \notin \{x_n\}$.

Limits

Let x_n be a sequence ($n \in \mathbb{N}$). Then, $\lim_{n\to\infty} x_n = x$ (i.e. x_n converges to x) if $\forall \epsilon>0$, $\exists N>0$ such that for all $n>N$ $|x_n - x|<\epsilon$.

When we are given a problem along the lines of "show that x_n converges", this requires that we know what the limit *is*. But once we know/are given what the sequence will converge to, there are a few ways to prove convergence:

- Using the definition of a limit directly.
- The squeeze theorem.

Recall that the squeeze theorem states that if given three sequences x_n, y_n, z_n such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L,$$

then $\lim_{n\to\infty} y_n = L$. The squeeze theorem is particularly useful if y_n has a complicated or unknown formula. Going back to our list, of ways to show a sequence converges:

- $\limsup x_n = \liminf x_n$ will show that the sequence converges.

In particular, proving the contrapositive can be useful for showing a sequence does not converge.

- Showing that x_n is a Cauchy sequence.

This is particularly useful if you do not know what the limit of the sequence is. Recall that a sequence if a Cauchy sequence if $\forall\epsilon>0$ there exists an N such that $|x_n - x_m|<\epsilon$ for all $m,n>M$.

Problem 7

Given that $x_n \to 5$ and $x_n \neq 0$ for all n, show that

$$\frac{1}{x_n} \to \frac{1}{5}$$

using the definition of the limit.

Proof: Where else to begin than with what we are given. We know that $x_n \to 5$, and thus for all $\epsilon>0$ there exists an N such that $|x_n - 5|<\epsilon$ for all $n>M$. Furthermore, we *want* to show that $\forall\epsilon'>0$, $\exists N'$ such that

$$\frac{1}{x_n} - \frac{1}{5} < \epsilon'.$$

To relate these two pieces of information, we can try to make the term $|x_n - 5|$ "appear" in the above inequality:

$$\frac{1}{x_n} - \frac{1}{5} = \frac{|5-x_n|}{|5x_n|} = \frac{|x_n - 5|}{|5x_n|}.$$

This would be really helpful if we can get rid of the x_n in the denominator, and to do this we can utilize the fact that $x_n \to 5$. Given $x_n \to 5$, there exists an N_1 such that for all $n>N_1$, $|x_n-5|<1 \implies 4 \leq x_n \leq 6$ for all $n>N_1$. Hence,

$$\frac{1}{6} \leq \frac{1}{x_n} \leq \frac{1}{4}$$

for $n > N_1$. Therefore, for $n > N_1$,

$$\left|\frac{1}{x_n} - \frac{1}{5}\right| = \frac{|5 - x_n|}{|5x_n|} = \frac{|x_n - 5|}{|5x_n|} \leq \frac{|x_n - 5|}{20}.$$

There are two ways to proceed here: the squeeze theorem, or still strictly using the definition of a limit. We will do both.

The Squeeze Theorem Approach: We know that

$$0 \leq \left|\frac{1}{x_n} - \frac{1}{5}\right| \leq \frac{|x_n - 5|}{20},$$

and the left and right hand sides go to 0. Thus,

$$\begin{aligned}
&\implies \lim_{n\to\infty} \left|\frac{1}{x_n} - \frac{1}{5}\right| = 0 \\
&\implies \lim_{n\to\infty} \left(\frac{1}{x_n} - \frac{1}{5}\right) = 0 \\
&\implies \lim_{n\to\infty} \frac{1}{x_n} \to \frac{1}{5}.
\end{aligned}$$

□

The Limits Only Approach: This approach involves ϵs and Ns but is still useful to know. We know that for all $\epsilon > 0$, $\exists N$ such that $|x_n - 5| < \epsilon$, Hence, let $\epsilon = 20\epsilon'$. Hence, theres an N_2 such that $|x_n - 5| < \epsilon = 20\epsilon'$. Let $N = \max N_1, N_2-$ thus it is true that for all $n > N$, $|x_n - 5| < 20\epsilon'$ and $\frac{1}{x_n} \leq \frac{1}{4}$. Hence,

$$\begin{aligned}
\left|\frac{1}{x_n} - \frac{1}{5}\right| &= \frac{|x_n - 5|}{|5x_n|} \\
&\leq \frac{|x_n - 5|}{20} \\
&< \frac{20\epsilon'}{20} = \epsilon'.
\end{aligned}$$

Therefore, for all $\epsilon' > 0$ there exists an N such that for all $n \geq N$,

$$\left|\frac{1}{x_n} - \frac{1}{5}\right| < \epsilon'.$$

□

Series

Here I will denote $\sum_{n\in\mathbb{N}} x_n$ (of a sequence x_n) as $\sum x_n$. In regards to series, there are two main ones to remember:

1. Geometric series:

$$\sum_{n\in\mathbb{N}} a^n = \begin{cases} \frac{1}{1-a}, & |a| < 1 \\ \text{diverges}, & |a| \geq 1. \end{cases}$$

2. The Harmonic series: $\sum \frac{1}{n}$ diverges. More generally,

$$\sum_{n\in\mathbb{N}} \frac{1}{n^\alpha} \implies \begin{cases} \text{converges}, & \alpha > 1 \\ \text{diverges}, & \alpha \leq 1. \end{cases}$$

These main ones are very useful to remember for the comparison test, which we will cover shortly.

There are a few ways to determine if a sum converges or diverges.

- The series is Cauchy. In other words, for all $\epsilon > 0$ there exists an N such that for all $n, m > M$

$$\left| \sum_{i=n}^{\infty} x_i - \sum_{i=m}^{\infty} x_i \right| = \left| \sum_{i=n}^{m} x_i \right| < \epsilon.$$

(Why does this imply the series converges?)

- Use algebraic operations on a series. Note that you have to be very careful with these sorts of operations. Let $\sum x_n$ and $\sum y_n$ converge. Then given constants a, b,

$$\sum (ax_n + by_n) = a \sum x_n + b \sum y_n,$$

hence implying the left hand side converges.

- If $x_n \geq 0$ and the sequence $\{\sum_{n=1}^{m} x_n\}_m$ is bounded, then $\sum x_n$ converges. This follows the Monotone Converge Theorem (how?).

Series Tests

Series tests give us a way to check if a series converges or diverges. There are two tests to know:

i) The Comparison Test. If $0 \leq x_n \leq y_n$ for all n, then: if y_n converges the x_n converges, and if x_n diverges then y_n diverges. This is generally true for all sequences (why?), and this is very helpful for series.

ii) The Ratio Test. Here, let

$$\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = L.$$

Then,

- If $L < 1$ the series converges.
- If $L > 1$ the series diverges.
- If $L = 1$, *we don't know.*

Lets look at two examples of series that we can use these tests for.

Problem 8

Does $\sum \frac{1}{n^2+1}$ converge?

Yes, using the comparison test. $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$ for all n, and $\sum \frac{1}{n^2}$ converges. (as stated in the previous section of this review).

Problem 9

Does

$$\sum \frac{n3^n}{(n^2+1)4^{n-1}}$$

converge?

Yes, and we can prove this using both tests (and we will).

Using the comparison test:

$$\begin{aligned}
0 \leq \frac{n3^n}{(n^2+1)4^{n-1}} \\
&= 4 \cdot \left(\frac{3}{4}\right)^n \cdot \frac{n}{n^2+1} \\
&= 4 \cdot \left(\frac{3}{4}\right)^n \cdot \frac{1}{n+\frac{1}{n}} \\
&\leq 4 \cdot \left(\frac{3}{4}\right)^n \cdot \frac{1}{1}.
\end{aligned}$$

Furthermore, $\sum \left(\frac{3}{4}\right)^n$ converges as it is a geometric series and $\frac{3}{4} < 1$.

Now we will use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{x_{n+1}}{x_n} &= \lim_{n\to\infty} \frac{\frac{n3^{n+1}}{((n+1)^2+1)4^n}}{\frac{n3^n}{(n^2+1)4^{n-1}}} \\
&= \lim_{n\to\infty} \frac{3}{4} \cdot \frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} \\
&= \frac{3}{4} < 1.
\end{aligned}$$

Hence, this series converges by the ratio test.

Note that for series of the form

$$\sum \frac{1}{n^\alpha},$$

The ratio test is not particularly helpful, as in sums of this form the limit of the ratio goes to 1.

Best of luck on your exam!