[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: OK. So let me first remind you about-- we'll be looking-- so last time we were looking at-- we had a sequence of functions on an interval I. And we have a function f on that same interval.

And then we talked about two notions of convergence. One was the pointwise convergence. So this meant that-- so we say that this sequence converges to f pointwise if, for all x fixed, you're looking at the sequence fn of x-- so x is fixed-- and that this here converges to f of x. So that was the pointwise convergence.

And the second convergence that we talked about was uniform convergence. So this is 2, uniform convergence. And so we say that this sequence here converges to f uniformly if, for all epsilon greater than 0, there exists an N so that if you're further out than N, then the difference between fn of x and f of x is smaller than epsilon. And this here has to hold for all x. And so, again, uniform refers to that it's for all x. So it's uniform in x.

And so there was one example that was particularly illuminating. And that was that we took-- so if I here was the integral from 0 to 1 and if we're looking at the function fn of x, that was just x to the power n. And f of x is then the function that is 0 unless-- so 0 if x here is between 0 and strictly less than 1, and 1 if x is equal to 1.

And so the picture was that-- so the picture was that you have this sequence here. They are defined on the interval from 0 to 1. They all end up at 1. And then as n becomes larger and larger, these functions become more and more steep towards 1, like that.

And so then we had that this sequence fn converges to f pointwise, because if you fixed x, then really these values here converge either to-- if x is not equal to 1, then as n becomes large, they converge to 0. If n is 1, then the fn of 1, that's always 1. So it's clear that it's converging pointwise.

But fn does not converge. Does not converge to f uniformly. And the reason why it didn't converge uniformly was that we saw that for each-- by the intermediate value theorem, if we were looking at 1/2 here, then for each of the function there would be an xn so that the value at xn was 1. Was a half, sorry. So you define xn so that the value of this is a half.

But now, if you're looking f of x minus f of xn like that and if this-- sorry-- is n, then this difference here is actually just a half. And this difference here should be small. If n was sufficiently large, it should be small for all x. But it's not. We've used one particular x to show that it wasn't.

OK. That was just a quick review of some of the things we talked about last time. Now, let's get to theorem. And so maybe also one more thing about this example is that you see that these functions here, these functions fn, they're all continuous. But the limit is not continuous. So pointwise. So note that-- so note that pointwise convergence is not enough to guarantee that limits of continuous functions-- that the limit of continuous function is continuous.

OK. So that's what this example illustrates. So now, instead, we have the following theorem. So we have this interval, and we have a sequence fn, and we have f all defined on this interval. And if we assume that all of these fn's are continuous, and b, we assume that the convergence here, the convergence of fn to f is uniform, then the claim is that then f here is also continuous.

OK. So let's try to prove that. So pointwise convergence is not enough to guarantee that the limit is continuous. Uniform convergence is enough. Now, let's see. So let's prove that. So given epsilon greater than 0, there exists capital N such that if little n is bigger or equal to capital N, then fn of dx minus f of x is smaller than epsilon over 3 for all x. This just follows that the sequence converge uniformly. So this is the first thing.

And the second thing is-- so this was, since fn converges to f uniformly. The second thing is that-- so now we have n. This here holds for all n bigger or equal to capital N. But in particular, it holds for capital N. So now, since f capital N is continuous-- so this is a fixed function now-- then for any fixed x0, we have that there exists a delta greater than 0, such that if x minus x0 is smaller than delta, then this function-- the value at these two points differ at most-- so this is for the fn. The value at these two points differ at most by epsilon over 3.

So again, we have a sequence, fn, that converges uniformly to some f. We want to prove that this f is continuous. So we just need to prove that for each x0 fixed, it's continuous at x0. First, I'm using that the sequence is converging uniformly. So given this epsilon, there exists this capital N so that if you're further out, then you have this. I'm not even going to use the part that is further out. I'm just going to pick this capital N.

And so then I'm using that this f capital N is continuous, and it's continuous at x0. So I have this. And so now we need to estimate-- we now need to estimate the difference. So what we're really interested is in f, but the f capital N, that's going to help us estimate the difference between these two numbers. And so these two numbers-- so x here, x0 was this fixed point that we want to show that f is continuous at that point.

And the delta, well, the delta came from f capital N. So we're looking at this x, this [? closed, ?] where the delta was given by f capital N. And now we use the triangle inequality. So this difference here is-- so you may as well think about this one here. And you could compare it with the value at the same point.

So I'm just writing down-- I'm writing this difference down as a sum of things. And then afterwards, I use the triangle inequality. So here, minus f capital N of x0. And then I'm adding back this here again f capital N of x0. And I'm comparing this with f of x0.

So you see that here, this is really the same because here I'm subtracting this thing that I didn't have, but I'm adding it back, subtracting this thing that I didn't have, but I'm adding it back. So I'm just left with those two things, which is that. And so now we use the triangle inequality, that this is now bounded by f of x minus fN of x plus fN of x minus f capital N of x0, like that, plus, and then f capital N of x0 minus f of x0, with an absolute value sign like that.

Now, this thing here is smaller than epsilon over 3. And that came from that it's evaluated at the same point and the convergence was uniform. So it really was the first part here. So that is less than epsilon over 3. This one here comes from that the difference between x and x0 is less than delta and that this function fN was continuous at x0.

So this is another epsilon over delta. This is this second part here. And then the last here is, again, the same that gave us this bound, namely that the convergence was uniform and is evaluated at the same point. So you have that. And so what we've proven here is that for this given x0, if the difference between x and x0 is less than delta, then the difference in the images is less than epsilon. So this means that the function f is continuous at x0 and x0 was arbitrary.

Now, the first application of this is that suppose you take-- suppose you're looking at the following example. xn of x is the sum here from k equal to 0 to n of xk over k factorial. Suppose you're looking at xn like this. Then the xn here of x, they are all continuous. They are sum of polynomials. They're just a polynomial, actually the sum of what's called homogeneous polynomial. But they are themselves polynomial. So they are all continuous.

And Weierstrass M-test. So by Weierstrass M-test that we proved last time-- by Weierstrass M-test, the convergence here to E, where E here is the exponential function. So it's this power series sum from k equal to 0 to infinity of x to the power k over k factorial. This convergence here is uniform. It's uniform assuming that you do it on a compact interval. So it's uniform on any fixed interval of the form, say, minus L to L.

So if you're fixing an interval like this, then the convergence is uniform. And so this means that since these guys here are continuous, then it implies that this one here is also continuous. So this is implied by the theorem that we just proved.

So now this will-- uniform convergence. There's another way of thinking about uniform convergence that, in a way, makes it more natural. So remember that if we looked at-- so if we take an interval-- let's say that this is an interval from a to b. a and b are real numbers. Then we looked at the space of continuous functions on this interval.

And this space had a natural metric. So I'm using here that both endpoints are included. This space here has a natural metric. And that metric is that if you take two functions-- two continuous functions-- on the interval, then the distance between these two functions is just the max of the difference between the two functions.

The difference-- again, why is this? The difference is continuous. So this means that the maximum is-- the difference is continuous. So this is a continuous function, and it's still continuous if you take absolute value. And so this means that the maximum by the extreme value theorem is achieved. And so this here was a natural distance between them.

Now to say that the sequence here-- to say that the sequence fn is converging uniformly, you can think about that in terms of this distance. So we have the following proposition. fn converge uniformly to f if and only if fn converges to f in the metric space with this metric.

Now, why is this the case? So this is just that-- so proof of that is really just a one-liner because you see that to say that-- so that f of x minus fn of x is less than or equal to epsilon for all x if this here is the case, where this is equivalent to saying that the distance between fn and an f is less than or equal to epsilon for all x.

Because if this is the case for all x, then the maximum is less than or equal to epsilon. This was the maximum. And on the other hand, if the maximum, well, then it holds for each x. So this is sort of a tautology. And so now we see, of course, that if the sequence-- so you have that.

But this means-- so this means that if the sequence-- therefore, if the sequence fn converges to f uniformly, well, then it means that for all epsilon, if we're sufficiently far out, then this here can be-- for any given epsilon, if you're sufficiently far out, this here can be made less than or equal to epsilon, which means that this error can be made less than or equal to epsilon if you're sufficiently far out. So you see that if this converges uniformly, then it implies that this thing here converges to 0. But that's exactly what it means, that this sequence fn converges to f in the metric space.

And the other way around. If this sequence converges, fn converges to f in the metric space, this is equivalent to saying that this here goes to 0. But if this here goes to 0, this means that if you're sufficiently far out-- for a given epsilon, if you're sufficiently far out, this here is less than or equal to epsilon. But it means that this here is less than epsilon. So that means it really converges uniformly.

OK. And that's in a way that makes it much more natural. I mean, yeah. So it makes uniform convergence much more natural. So now let's prove the next term. So for this theorem, we're going to think about uniform convergence in terms of this metric and in terms of convergence in this metric space. So this theorem is the next thing.

Let's say that if you take an interval from a to b and you're looking at the continuous function and you equip it with this metric that we talked about, then this here is this space, which is a metric space. So this metric space is a Cauchy complete. So remember, Cauchy complete in a metric space means that if you take a Cauchy sequence-- so just let me just recall here over here.

So Cauchy. So if x comma d is a metric space, then Cauchy complete means that any Cauchy sequence is convergent. That's what it means to be a Cauchy complete. And of course, an example of a Cauchy complete metric space was just if we took R with the usual metric.

So what we have here is-- so let's try to prove this. So we need to prove. So this here is now-- we need to prove that if this here is a Cauchy sequence, then there exists a function that is also continuous. And so this sequence here converging to f in the metric d. That's what we need to prove.

So in particular, we need to first construct a function, f. So now let's try first to construct a function. So we first construct f as follows. Well, if you fix x-- so x is something, a point in the interval.

And now look at this sequence fn. This is now a regular sequence. But if you're looking at the sequence, then we have that fn of x minus fm of x, if you have n and m like this, then the difference here is less than or equal to the max where this is for all z in the interval of fn of z minus fm of z, like that.

Because of course, it's like in particular if z was x. So you have clearly this. But now you see that-- so I claim that this means-- so I claim that this implies that this sequence here-- sorry, this sequence here. fn of x, this sequence here, which is a sequence of real numbers-- these here are real numbers-- that this here is a Cauchy sequence of real numbers.

Why is that the case? Well, it's the case because given-- so this is just because given-- so this is the case because given epsilon greater than 0, since fn is a Cauchy sequence in the metric space-- in this metric space-- then there exist capital N so that if little n and little m are bigger than capital N, then the difference between these guys-- sorry, the difference between this function here and that function is less than epsilon.

This is because they were Cauchy sequence. And so now this thing here is, of course-- this thing here, that's what we just saw. This is the maximum. So this is bigger than when you evaluate it in just one point, like that. And so this proof that when x is fixed, these guys here is a Cauchy sequence of real numbers.

So now we use-- since R here is Cauchy complete and fn of x is a Cauchy sequence, then it means that the sequence fn of x, that this is converging, and it's converging to some number, to just one number. And that number we can call f of x. So we just defined f to be the limit of the sequence. And that's well defined because there's just one number. So for each x it gives you a number.

OK. So now what we have proven is-- so now we have a candidate. So namely, we have shown that-- so now what we have proven here is that we have found a function. So we have now proven that this sequence here is converging to some function f. The convergence is just pointwise.

And we don't know whether this function is continuous or not. The convergence is pointwise. This is just because for each fixed x, f of x was defined so that it was a limit of these guys, fn of x. So now we need to promote this convergence that was pointwise, and we need to promote the convergence to uniform convergence.

So now we need-- so given epsilon greater than 0, again, since these guys, fn, was a Cauchy sequence in the metric space-- in this metric space-- then as we saw over there, there exists capital N so that if little n and little m are bigger than this capital N, then the difference between these two, fn of x minus fm of x, this thing here, is less than or equal to-- and I'm using epsilon over 2-- for all x. So you have that.

But now this here holds for all m and n. Let's just fix n. And now we can use that. It's still true. I can still let m vary. And so if you let m go-- so fixed n bigger than capital N. And let little m go to infinity.

Well, this has just some fixed number. But as m goes to infinity, this here goes to f of x. And so since this difference is always less than or equal to epsilon over 2, it's also the case that the difference here of the limit is less than or equal to epsilon over 2. So you have this here for all x. This is just letting m go through to the limit.

But this, of course, says that the max, if you will-- so you have that. So this implies that the max of fn of x minus f of x, that this thing here, where x is now in this interval, that this is less than or equal to epsilon. But this is the same. This here is just the distance between fn and f. So say that this distance here is less than-- I guess I had epsilon over 2. So it means that this thing here is strictly less than epsilon.

It's actually less than or equal to epsilon over 2. So it's strictly less than epsilon. But that's exactly what it means for this sequence here. So this is the same. This is the same as saying that fn converges to f uniformly. Sorry, here, i should have taken the sup. The reason why I should take the sup here is that f is just a function. It's not necessarily continuous at this stage.

But it was clearly the case with the sup. So the maximum a priori may not be achieved yet. But now you see that they converge. These fn converge to f uniformly. And these guys here was continuous. These are continuous. So this implies that by this theorem we just proved, that f is also continuous. And so you see that-- so now you have that the limit is indeed in the space. And we already proved here that the convergence to this limit is uniform, which means that it's in the metric.

The next thing is that-- so this here was just a little bit different way of thinking about uniform convergence, that you think about it more abstractly as a convergence in a metric space. And it's a useful way of thinking about it. The next thing we want to look at is how this uniform convergence gives you a way of computing things. So in particular, if you have some power series.

And we talked about that a little bit last time. We didn't prove anything. But one of the motivation for this is that suppose you take a power series, like this. And we'll return to this in just a second. But suppose you take a power series, like this, and you are inside the radius of convergence. So inside radius of convergence. So you know that this thing here makes sense, that this, for each x, gives you a number.

And so it gives you a function. And so you're interested in, what is the derivative of that function? And what is the integral of that function? How do we integrate it, and how do we take the derivative?

And the thing that worries you a bit is that if it was just a finite sum, it would be trivial. But it's an infinite sum. And so is it OK-- so the natural question is, is it OK? If you want to take the derivative, can you interchange this infinite sum with taking derivative? And likewise, if you want to integrate this infinite sum, is it OK to interchange the infinite sum taking the integral?

And the answer is, it's OK. So let's see that. So we do this. So we have two versions of this, one that's dealing with taking integrals and another version that deals with taking derivatives. And so the first thing is that I take-- so the first theorem is the following, that I have some interval from a to b. And then I have a sequence of functions, fn, that are integrable. They may not be continuous, they're just integrable.

So in particular, they are bounded. And then I have that fn here converges to f uniformly. And I want to prove that-- so the claim is that fn is also integrable. And the integral of f is equal to the limit of these integrals. Sorry, ir particular, of course, those integrals here has a limit. It's part of the statement.

OK. Right. So what do we want to-- what do we need to prove? So prove. So we want to show-- so we first want to show that f is integrable.

And we had our criteria for doing that. So we're using the criteria. So we can use that all we need to show-- to show-- is the following, that given epsilon greater than 0, there exists a partition P of the interval from a to b, so that the upper sum of f with respect to this partition minus the lower sum of f with respect to the same partition, that this thing here is less than epsilon.

That's all we need to prove. We need to prove it. For each epsilon, we need to prove that there exists a partition. So if we have that, then we already know that then f is integrable. So that's just achieved this first task. And then, afterwards, we need to prove that.

Right. OK. So now, how do we prove that? So the first thing we observe is that if we take-- if we're looking at-- suppose that fn and f-- here. Suppose that fn of x minus f of x, suppose that this thing here was less than epsilon over 3 b minus a. Suppose you have that. And of course, this here can be achieved. So epsilon is now given. And so you have that.

So maybe I should just say that. So epsilon is given. So since fn converges to f uniformly, this means that there exists capital N so that if you're further out than capital N, then actually you have this thing here. b minus b. b and a are fixed. This is just something fixed. So this is just another epsilon.

OK. So if you have that, now suppose you take-- so remember, we defined, when we did the-- when we had the partition, the partition had the dividing point with dividing points xi. So we were looking at these small intervals from xi minus 1 to xi. And we're looking at the sup of the function.

So if you're looking at the sup of-- suppose you're looking at-- so this here is on this little interval. But it's for f. But the function f here, you see the function. So you have that f of x, this function here-- sorry, I don't even need the absolute value. But f of x is less than or equal to fn of x plus, of course, f of x minus fn of x.

And so this means that this thing here is-- this thing here is less than or equal to fn of x. This thing here is bounded by the absolute value of it. And so this thing here-- so this is fn of x. And this here is just epsilon over 3 b minus a, like that.

So this means that if I take the sup on this little interval of f, then it's bounded by the sup of this function plus that number. So I have that-- so I have that, again, the sup of f on this little interval-- this is what we call Mi of f-- that this is less or equal to-- because of this inequality, it's less than the sup of fn of x on the same little interval plus epsilon over 3 b minus a.

And likewise-- so likewise for the inf. So again, I let little mi of f, this is now the infimum on this little interval. Oops. On this little interval of f. Sorry. Here, I wanted to insert what this thing was. So this was M. The notation for this sup here is Mifn.

And the infimum for f on this little interval, the notation for that is this little m, i referring to the interval, f to the function. And so similarly, if you take the infimum, and you compare it to the infimum of this fn on the same interval, then the same argument with obvious changes means that mif here is now bigger or equal to mifn minus epsilon over 3 b minus a.

It's the same-- OK, maybe I'll just explain it, but it's the same sort of thing, that you take-- oh, maybe I shouldn't erase this one. So it's sort of the same thing, that you're looking at-- you have f of x again is equal to fn of x plus fn plus f of x minus. So you have obviously this.

And so this means that f of x is bigger or equal to fn of x minus. If you think that this here becomes negative and the maximum it can become negative on this little interval here-- on this little interval, the worst this here can become negative is epsilon over-- actually, on any interval, it's this. Like that. So you always have this.

OK. And so if you take now the infimum on this little interval, then you're getting mif bigger or equal to mifn minus epsilon over 3 b minus a. So you have that. So that's what we claim over there.

OK. So we have this thing here, and we have the corresponding thing. This is less than or equal to the maximum of the function on the same interval. And then that's-- and then this equality here, this equality here was that it's bounded by this guy here plus epsilon over 3 b minus a.

So this is for the little m's, the little mi's, and the capital Mi. And now we need to sum these guys, and we need to multiply those with a delta xi. So this means that if you're looking at-- so what we're interested in is we're looking at this times delta xi. This gives you the-- sorry, this gives you the lower sum of f with respect to this partition, where you're summing here over the i. If you're summing the sups, then you're getting the upper sum, like that.

And so now-- so you take-- you take this inequality, this string of inequality, and you're just multiplying by delta xi on both sides. And then I'm summing. So if I multiply-- so let's call this here star. So multiply star by delta xi and sum over i. If you do that, then what you get is you're getting mifn minus epsilon over 3 b minus a times this, times delta xi.

This is less than or equal to mif delta xi, which is less than or equal to this one here. Sorry. And I should also sum. Summing over i, like this. And then I have one more inequality, namely this right-hand side. So less than or equal to. I'm summing over i. And I have-- this is mif plus epsilon over 3 b minus a times delta xi.

So we have that. Now let's try to figure out what this means. So this here-- so now, if you multiply this out, the first thing, the top left corner, that gives you-- when you multiply the bracket out, the first term you get is the mifn times delta xi. But that is just the lower sum with respect to the partition P for fn.

But then you get another term, which is you get one other term. You get this minus. And then you're summing over the i of epsilon 3 b minus a times xi. And then it's less than or equal to the term up there. That's just the lower sum of f with respect to the partition. THE next term is the upper sum of f with respect to the partition. And then the last inequality, the first term-- when you multiply this bracket out, the first term here is-- and unfortunately, there was a missing n. I copied it from here. Sorry. There was a missing n. I just copied it from this one.

So now when you multiply this bracket out, the first term you get here is-- so you're getting less than or equal to the first term you get there when you're summing over the i. That's exactly u fn, p. And then the second term you get is this i epsilon over 3 b minus a delta xi.

Now, this thing here, when I'm summing here, then this is just a constant. So I can just pull that constant out. And the sum of these differences, that just gives you b minus a. So if you put that in and it's exactly the same you get on this term, then you have that L fn, P minus epsilon over 3 is less than or equal to L f, P, which is less than or equal to u f, P. And this is less than or equal to u fn, P plus epsilon over 3.

OK. Now what have we used so far? So far, we have just used-- P was any partition. And all we have used is that the fn converges to f uniformly. And then, for this given epsilon, we chose this capital N sufficiently large. But now, now you see that-- now we can just fix-- so now we just let little n-- we just said, let this be equal to this capital N. We just need to use it for one little n.

And then we can-- and now we have that since this particular function, this one function, was integrable, then there exists a partition. But this is now a partition that comes from, that is determined from this function here. But there exists a partition so that the L-- sorry, so that the difference between the upper sum for this function minus the lower sum for this function is less than, say, epsilon over 3.

So you see that this number here-- so you see that the difference-- now I can erase this thing here. So now the first-- so we observe now that this number here, because of this, this number is always larger than that. And so this means that this thing here, by this inequality, it just tells you that this is less than-- that this thing here-- let me continue here below-- that this thing here is less than L fn, P plus epsilon over 3.

This is coming from that this one here-- just moving this over on the other side. Then you have that this is bounded by that. And then you have another epsilon over 3. And so you see that those two things-- this here and that here-- is squeezed in an interval of length epsilon. So this means that the difference between these two is bounded by epsilon.

So L f with respect to this partition and u f, P, they both lie in an interval of length epsilon. So it means that the difference between the two. So the difference between. So this here, which is always positive. But this here, the difference must be, I guess, less than or equal to. It doesn't matter.

And this proves-- so this show that f here is integrable. It's integrable. Because to show that this function was integrable, we just needed to-- for a given epsilon, we needed to find a partition so that the difference between the upper and the lower sum was less than or equal to epsilon. So the function f is integrable.

Now, I claim that it also proves that-- it also shows that the limit is what it should be. Because you see that we now know that the function-- so we have now that this integral of f dx, this exists, because we proved that it's integrable. So it means that this exists. And this is always squeezed between-- for any partition, it's always squeezed between these two things.

But now you see that-- so this means that it would have to lie-- in this string of inequality, it would have to lie here, between these two numbers. But this thing here, if you make this-- so we have also-- so what do we have? We have that-- so we have that this integral here, from a to b of f dx, is less than or equal to u f with respect to any partition. The same thing here.

And now, if we let-- if we looked at-- if we use this-- if we use, for this given epsilon, that n here is bigger than capital N, bigger or equal to, then we have that this thing here was bounded below from this inequality L fn, P minus epsilon over 3. And we had that it was bounded from above. So this is just this inequality u fn, P plus epsilon over 3.

And so now we made use of-- and so we do for this capital N. So in particular, it holds for capital N. And now we made use of that. So now all I want to do is here I want to estimate the difference between this integral here and this integral here. And the integral from a to b of f capital N. And I want to prove that if I'm very far out, that this here is very small.

But I now make this one here-- so if I take m-- when I choose my partition so that this number here-- so then I can just make sure that this number here, when I choose the partition, then I just said that I choose the partition so that this thing here minus this lower here for this capital N, that this here was smaller than epsilon over 3.

And so this means again-- but now you see that this integral of this integral here, this is always squeezed between these two numbers. So this is always squeezed like this. And so now you see that this means that this integral is in this interval here. But in fact, of course, in this interval between these two numbers-- between these two numbers-- even if there wasn't a plus in that interval, that this integral here would also lie.

So it lies even more comfortable in this larger interval, where you're adding epsilon over 3 and subtracting epsilon over 3. So this integral lie between these two. The same is the case for this. And this interval, I can have length epsilon. So this means that this difference between these two numbers is also less than epsilon.

And so this really proves that if it was sufficiently far out, then the difference between these two integrals was bounded by epsilon. So this proves that the integrals are converging. OK?

Let's look at-- so how do we use this? So what we were interested in was using this fact in the case of a power series. So let me just look at an example for that. So suppose that En of x-- so this is the same as in the previous example, where I'm summing here from k equal to 0 to n xn over-- xk, sorry, over k factorial. So these are the functions xn and x-- sorry, En.

And the function-- so this was the function En. The function E is just the exponential function, which I think about as the infinite series, power series, like this. And now I had-- so we already observed that by the Weierstrass. So by Weierstrass M-test. By Weierstrass M-test, the En converges to E on any interval. You have to restrict the interval. But if you're fixing an interval like this, then it's converging uniformly. So this is what Weierstrass M-test gave.

And now these functions here, they are polynomials. The En are polynomials. So it's easy to integrate them. So you have that the integral here of En dx from, let's say, some interval from a to b, this thing here is now equal to the sum here. So now it's just a finite sum, so you can exchange that. And then you're integrating from a to b of xk over k factorial.

And of course, that's very easy to do. So dx. And the theorem that we just proved is that this thing here, because the convergence is uniform, each of these is integrable. This means that the limit here is also integrable, and the integral of the exponential function is the limit of these integrals. And so the point here, again, is that if you have uniform convergence, then you can-- I mean, you're using the uniform convergence that we have from Weierstrass M-test. Then we can interchange this infinite sum with taking integrals. OK?

OK. Let me just-- I had wanted to do one more thing, but I don't have time to do that quite. So I'll pick up on that next time. But let me just observe that when you're integrating here, when you're integrating this thing here, what is the integral of this? So when we just calculating this. So you're integrating x to the k over k factorial. When you're integrating a polynomial, you're looking for-- you're using the fundamental theorem of calculus. So you're trying to look for a function whose derivative is this.

Such a function is often called the antiderivative. And so what is that function? So a function-- so if you're looking at x to the power k plus 1, you know that you have to-- because it's polynomial, you have to add 1 to the exponent. And then, when you take derivatives, this comes down. So you have to look at k plus 1 factorial. Because when you take the derivative, then k plus 1 comes down. And it kills the first one, and the rest is this.

And so when you evaluate this integral, then this here is the antiderivative. And you have to insert the boundary values like this. So that would be the integral. So I'll just stop here. And next time, I'll go through the one theorem I didn't get to, which is how to interchange the differentiation.

Well, if you take a sequence of functions that converge uniformly and you want to take the derivative of these, how do you relate the derivative of the sequence to the derivative of the limit? But it's in the notes. I already posted the notes, but I'll go through that next time. Any questions? Yeah?

AUDIENCE: I had a question about the upper-number sums.

TOBIAS COLDING: Yeah.

AUDIENCE: I think there was a-- [INAUDIBLE] I've seen lot of difference between epsilon and numbers less than or equal to.

TOBIAS COLDING: Yeah.

AUDIENCE: Is that OK?

TOBIAS COLDING: That's OK because-- I'm not 100% sure I needed epsilon less than or equal to epsilon. It doesn't really matter because you just pick that. I had chosen it with the epsilon over 3 and epsilon over 3 times b minus a. That was just to have it worked out so it was epsilon in the end. But you could, of course, have chosen it even smaller. Then you would have had strict inequality. It's possible it was actually strict inequality. Anyway. OK