[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: OK, so last time we were looking at a metric space. So this is a metric space. And we looked at what we called a open subset. And so an open subset is-- so over here is now a subset of x is said to be open if for all x in O there exists r greater than 0 so that the entire ball around x is contained in O.

And so the picture is that here you have x. Here you have some open subset O. And so if for each point x in O, there is an entire ball here that is contained in O, then O is said to be open.

One of a few things that we proved is-- so one thing we proved is that-- so again, x, d is a metric space. Sorry, I should say we defined the ball, Br of x. So here, x is in x. r is some positive number.

And then this thing here is then the y in x such that the distance from x to y is strictly less than r. And we showed that Br of x is given. By that Br, we say also that Br of x is a ball of radius r centered at x.

Now, there's a few other things that we discussed. So one thing is that so on subsets, we have three operations. One is intersection.

So this is that if u alpha is a family of subsets, then we look at the intersection of all these subsets. And so x is in this intersection just means that x is in u alpha for all alpha.

The second operation we looked at was union. And so again, u alpha is a family of subsets. Then we say that x here is in the union of the u alpha. This is the same as saying that there exists alpha such that x is in u alpha.

And the third thing we-- the celebration we have is that if you have a subset capital U, then we have the complement. So the complement of U is the x such that x is not in U. That's a complement. And so there was these three operations on subsets.

And why this is relevant here? This is relevant because if u alpha-- so we have the following lemma. Again, we have a metric space here, and we have O alpha is a family of open subset of x. Then the union is also open. Now why is this the case?

Well, it's the case because if x here is in the union of the O alpha, then this means that there exists alpha such that x here is in alpha. But since that alpha is open, this means that there exists-- so not only is x in that, but there exists r that depends on x and alpha r positive, so that the whole ball centered at x is contained in O alpha.

But if it's contained in alpha then this imply-- if it's contained in alpha, then this implies that it's, of course, this ball here is contained in the union of these guys. But that's exactly what it means, to say that the union is open.

The other lemma related to this is-- and there, you have to be careful that if you have finitely many. So can you have a metric space and O1 up to On finitely many open subsets? If you have finitely many open subsets, then the intersection of these is also open.

Now why is that the case? This is the case because if you have-- so if x here is in the intersection of these guys, then it means that this is equivalent to saying that x here is in z for all i-- i equal to 1 up to n.

But since Oi is open, then for Oi, there exists ri. This means that since Oi is open, there exists at ri such that the ball here with radius r but center x, this one here is contained in Oi. This is for each i.

But now if we check-- there's only finitely many i so you can let r be the minimum of these guys. If r is the minimum of these guys, then the ball of radius r around x-- these are the points where the distance is strictly less than r to x. But that is, of course, contained in Br i of x. This is for each i, you have this. But this one here was supposed to be contained in Oi.

And so you see that this ball here-- again, this ball here is contained in this for each i. So this means that this ball must be contained in the intersection of these guys. And I'll just write it out again, just emphasizing that I'm only doing this for finitely many Os.

And that's exactly what it means, that the intersection is open. And again, you could draw a little picture here. It might be helpful to draw a little picture. You have this here is O1. Here, you have O2. And here, you have O3. You have a point in the intersection. There's a ball here that is contained in this.

There's another ball here that's contained in this. And there's a third ball, this ball here. So that's contained in that. And then you look at the smallest ball. You look at the smallest weight-- the minimum, again, of the radii. And that ball is then contained in the intersection of the 3.

Now why do we have to be careful? Let me just remind you, why do we have to be careful here? And so the examples that we looked at last time was that if you are on-- so the example from last time is that if x here is the real number and you're looking at On-- n here is a natural number-- and you're looking at this interval here from minus 1 over n to n, well, then if you're looking at the intersection of all of these, well, that's just the one point.

And of course, 0 is not an open subset. Because if it was an open subset, a whole ball-- and a ball here on the line is like a little open little interval that doesn't have the two endpoints. So note that in this example, each Oi or each On is open, but the intersection of the On's is not open. And the problem, again, is that there are infinitely many of n's.

Now again, one of the things we proved was-- and maybe I was just here on that board-- but one of the things we proved last time was that if you take a ball, then it is actually an open set. Let me prove a variation of that.

So suppose you're looking at-- so let's have a lemma here that if you have, again, a metric space here and r here is bigger than 0. Now let's look at the x and then take away. But now I'm looking at the sets. So let me look at the set y here. So x is some fixed point in this metric space, and you're looking at the y in x so that the distance to x from y is strictly bigger than r.

So in other words, if this here is x, this is the space x. Here, you have the point x. Then you have here this, including this circle here, are the points that have distance at most r from little x. Then this is the set here outside this disk, where the boundary is included, so outside.

And so what I want to prove here is that this here is this here is an open subset. And the proof is very similar to how we proved that a ball is open. Again, the key ingredient is the triangle inequality. And so let's try to prove that. So we take y here in this-- let's call this set something. Let's call it E.

So suppose you have x here, y here in E. So this means that the distance from y to x is strictly bigger than r. I want to prove-- so I have my picture here. This is x. This is this ball here. And then I have a y that lie outside it. And I want to find a little ball here around y that is entirely contained in the complement of this ball.

And so I do the same that I did last time-- I mean, almost the same, not exactly the same. So we said S here equal to y. So the idea is that you're looking at this distance here. This distance is d x, y. This part of it is r. And so it seems reasonable, this here, if this picture was right, then this thing here would be s. And this s here would be d x, y minus r. So that's what I'm setting s equal to. And the first thing I noticed that because this is strictly bigger than r, this thing here is strictly positive.

And then I take set here in that ball. Now it's centered at y. Let's have radius s. And then I want to look at the distance to-- and I want to estimate. So I want to estimate from below the distance from x to z. Well, the triangle inequality, if you just think about the triangle-- that's something like bounding a distance from above. So if I want to estimate something below, then basically, this thing I want to estimate from below have to be on the other side of the inequality.

And so how does that work? So let's again draw a little picture, too. So let me do a little bit on the side. So I have this ball here, and I have this y here, and then I have this z here. And I now want to say-- I want to estimate. So I'm saying that the distance from-- so this is what I want to bound from below. And what I know is that this set here is-- I know something about the distance to y. So this here, I know.

And so you see, this thing here by the triangle inequality is now bounded by the distance from x to y. And I know that this thing here is less than or equal to this here is the one I want to estimate. So I'm just leaving that. This one here, it's because set is in the ball of radius s around y.

So this thing here is actually strictly less than s, like that. And so now you can just move this one here over on the other side. And you have that d x, y minus s is strictly less than d x comma z.

Now we have to fill in what S is. S is up there. So this thing here is-- this thing here is equal to this is dx. I'm just copying that. But I fill in what s is, and s is d x, y minus r. But you see, those two cancel, and you're left with the r. And so you have that r is strictly less than this, which is exactly what we wanted to prove.

And so, again, it was almost exactly the same proof that we had to prove that the ball was open. What we're using here is that, again, we're using that the triangle inequality is like you're bounding some distance from above. And so you have to use it the other way with having the things that you want to bound from below on the other side of the inequality.

So the next thing is then-- and we talked a little bit about that last time-- the next thing is closed subsets. So this may seem like a triviality because it's defined, in a way, trivially. So what is it? But it's actually a really useful notion. And so this is-- so a subset C is said to be closed-- again, x is a metric space-- to be closed if the complement is open. So that's how you define a closed subset.

And so, again, as we observed, since the empty set and x are both open, the complement of the empty set is the whole set. So it means that x and the complement of the whole set is the empty set. It means that those sets are both open-- are also both closed.

So now there is another way-- you could have defined a closed subset another way. And this is because you have the following statement here. So a subset-- you have the following theorem that, again, you have a metric space.

And C here, subset of x, is closed if and only if for all sequences in C-- if and only if for all sequences xn where xn is in C, if you have all sequences, all, let's say, convergent sequences-- so all convergent sequence where the sequence lie in C, then the limit x is also in C.

So I claim that if you want to check that a set is closed, instead of checking that the complement is open, you could check that. And this seems like if you were literally going to check it on any given example, explicit example, you would never do it this way. But it's very useful for proving things.

And so the thing is that instead of proving that a complement is open, instead you could just prove that whenever you have a sequence in that set, then is the limit also in that set? If that's the case for all sequences, then the set is closed. And so we need to-- so let's try to prove that. And again, it's useful theoretically. But in practice, this would be totally useless. But it is useful to proving things.

And so there's two things we need to prove. So there's two directions. So assume first that C is closed. And let xn in C be a convergent sequence. We want to show that the limit x is also in C.

Since xn converges to x, then for all r greater than 0, there exists xn in the ball here of radius r around x because it's converging. So this means that for each r, you must have this.

But now you see that. And if x was not in C, if x is not in C, since c is closed, the complement is open. And so this means that now x is assumed to be in a complement. But that means that there exists some r so that the ball here of radius of that r would be contained in the complement.

But this means that there is no xn in that ball because all of the xn's are in C. And this here is entirely contained in the complement. So this is the contradiction. So if it is closed and you have a convergent sequence, that sequence is lying in this closed set, then the limit must be also contained in the set because, otherwise, there would be-- the limit would lie outside, and there would be a little ball centered at that. So there was no element in the sequence that was in that ball. But then that sequence couldn't converge to the center.

And the other thing-- so this is one direction. The other direction that we need to prove is that if-- again, it's useless. You would never go about to prove that a set is closed in an example, but it is useful more theoretically. So suppose you have a set, and you knew that whenever you have a sequence in that set, then the limit that is convergent-- then the limit is also in the set. So we also need to show the converse and, again, the converse.

So we need to show that if for all sequences xn with line C n is convergent, then the limit x is also in C. Then C is in fact closed. So that would be the converse. So we prove if and only if it has this property.

So now let's assume that-- so suppose we have this property that all convergent sequences that the limit convergent sequence in the set. Then the limit was also in the set. Now let's prove that the set is closed. So to prove the set is closed, we need to show that the complement is open. So we need to show that the complement here is open.

So take x here in the complement. If for all r greater than 0-- so we need to prove that-- sorry, let me say it like this. So we take some x, and we need to show that there exists some r strictly bigger than 0 so that this ball here is contained in the complement. That's what we need to prove. And we need to prove this for each x in the complement.

Now suppose that if this was not the case, then I can look at for each n-- then for each natural number here, you can look at the ball. It's always going to be centered at this x. But I'm looking at it with radii that is smaller and smaller. If for each n, natural number, if that ball was not contained in the complement-- so this means that for each n, you could pick-- so pick xn in that set. So you just choose.

Because we're assuming that the complement is not closed. It's not open. So it means that there would exist at x so that no ball here would be entirely in the complement. So this means that this one here would not be entirely contained in complement. So this means that there would be some xn. And this xn here would actually be in the set C.

But now, of course, the distance here from xn to x, this distance here is strictly less than 1 over n. So this means that-- so this implies that the xn converges to x. And now the assumption was that whenever you have a sequence that was converging, these guys lie in C. Then the limit here also lie in C. So then you have, from the assumption that x here is in C. But this is the contradiction. You're assuming that x was not in C.

And again, this here is really, really useful. But it's only useful of as a tool to proving things. It's not useful for checking whether an example is open or closed.

Let me just quickly give you a couple of examples. I probably should have done that already that. If you're looking at-- so what are closed subsets? So closed subset could be-- so here's some examples-- closed subsets.

So if it's an r, a closed subset is some interval, let's say, from 0 to 1, where both these points are-- both 0 and 1 is included. This has closed. And why is it closed? Because the complement here is open. It's easy to see that the complement is open. Something like if you include one of the endpoints but not the other, this is either closed or open.

And also, if you're looking at the complement, say, from 0 of the complement of-- so if you're looking at this thing here, union, we'll say, from 2 to 3, this is also closed.

And maybe one more example is if you just take a single point, this is closed. And again, you just check that the complement is open.

So now the next thing, which I think we talked very briefly about that last time also, because we talked a little bit about closed subset. So you have the following theorem. So if you have a metric space here and you have a family C alpha of closed subset, then the intersection of these closed subset, this here, is also closed.

And if you have just finitely many closed subset, then the union of finitely many closed subset is also closed. And so here, it's exactly the opposite thing as for open sets that you can take any intersection, and it's still closed. But if you take union, you can only do it for finitely many. So this is exactly the opposite. Why is it exactly the opposite? It's because in order to prove that something is closed, it means that the complement is open.

And then we have these basic relations for-- so the proof of this again. And so these are these operations on set that we went through, that if you take x here and you take the complement of intersections of subsets. This thing here is the same as the union of the complement, like that.

And you see that since this is closed, this means that the complement is open. And you can take any union of open sets, and it's still open. So that's why you could take any intersection.

And so the other thing is that if you're looking to prove that this thing here is closed, then you're looking at the complement of this union here. But this thing here is now the intersection, the complement of the intersection of the C1 up to-- sorry, this is the complement of-- this is the intersection of these guys, like this.

And you see here, since C1 up to Cn, since they are closed, these guys here are open. And you're looking at finite intersections of open sets, and that's open. So again, for closed sets, in terms of taking intersection and union, it's exactly the opposite.

Now, the next thing I want to talk about is a cover. So have a metric space. So we have a metric space here. And we have a subset-- A here is a subset of x. And then we have a collection.

And then you alpha is a family of subsets of x. And we say that u alpha is a cover of A if A is contained in the union of the u alphas. So this is what it means to cover that set by a family.

A subcover of the cover u alpha is-- so imagine that you have a whole bunch of sets. And together, they cover A sub cover means that you don't need to take all of the u alphas, if you can just take some of them, and they will still cover it. So a cover is a subfamily of the given family u alpha such that the union of those still cover.

So let's look at a couple of examples. I mean, if you take a family that covers something, then the family itself is a subcover. But that's a kind of trivial subcover. And so you could look at-- so just here's an example. Suppose that the metric space is x with the usual as r-- sorry, with the usual metric. And suppose now that the u alpha is actually Un's, and those are integral from minus n-- let me just make them open. It doesn't matter. For a cover, there's no assertion whether they're open or closed or neither.

So if you're looking at those sets, well, if you take the union of all those sets, that gives you everything. So this family is a cover of the real line. Of course, if you take a proper subcover, meaning-- I mean, well, you could take a-- so this here cover all of it. But you see that this particular cover has the property that Un plus 1-- this is contained, the previous one.

So you don't really need all of them to cover r. If you wanted to, you could start with some very large n, and then you could go on from there. But there wouldn't be finitely many of the Un that would cover. You need infinite money to cover because, otherwise, there would be some of last point and both interest.

So that's a cover and a subcover. And an open cover is, then, where each u alpha is an open subset. I mean, you it's really distinct that-- so that's our open cover. So each of the sets that is in the covering has to be open.

And the last definition we need is a finite subcover is a subcover consisting of finite sets. So that's a finite subcover. And so now this leads to the-- and again, in this example here, r where this here is a cover, there is no finite-- so note that this particular cover does not have a finite subcover.

If you just take finitely many of these, they are not going to cover all of r. But actually, in this example, you see also that the cover is an open cover. So this now leads to the definition of compactness. So you have a metric space. So x comma d is a metric space. A here is a subset. And we say that A is compact if for every open cover of A there is a finite subcover.

And the thing that is really useful is-- and we won't prove this today, but I'll prove it next time. So this is the following result. So why is this notion useful? So one of the key properties, when you're thinking about sequences, one of the key properties for sequences on the line was the Cauchy convergence theorem.

And the Cauchy convergence theorem, again, said that whenever you have a Cauchy sequence, then it was automatically convergent, and it was often much easier to check that a sequence was a Cauchy sequence than to actually find the limit and prove that it's converging to that.

But to prove the Cauchy convergence theorem, we needed the Bolzano-Weierstrass theorem. And so this is because-- so why is it useful, this notion of compactness? So it's useful because there's a metric version of the Bolzano-Weierstrass theorem for metric spaces.

And so this is the following theorem. And again, I won't prove it now, but I'll prove it next time, that if you take a metric space, and A here is a subset and A is compact, then any sequence in A has a convergent subsequence.

So remember that the Bolzano-Weierstrass theorem, again, said that if you take an upper bounded sequence in r, then there is a subsequence that converge.

So now let's prove a few things about-- and let me also look at a couple of more examples. So the first thing I want to just discuss is that-- so this here is the key-- this is our key application of compactness that you have this metric version of the Bolzano-Weierstrass theorem. But let's prove a couple of things about compact subsets. So we have a metric space here. And I have A here is compact. Then I want to show that this implies two things. One is that A is bounded. And the second one is that A is closed.

Let's prove first that A is bounded. So you could look at a very trivial cover of A. So you have here A. Well, obviously, if you take all balls, let's say, with radius 1, and you take all balls, but now the center here could be anything in A.

For each element in A, you take a ball of radius 1 around that point. But first of all, these balls are all open. And of course, this thing here, the union of these guys, well, the union contains, in particular, all the centers, but the centers is already A. So you trivially have this.

Now since A is compact, this means that A must be that this is an open cover. So it means that finitely many of them already cover A, so you have finitely many, like this.

But now I claim that this automatically implies that A is bounded because, you see, you have all these balls here. And A is somehow contained in the union of these guys. So this means that for any element-- so this means that for any x-- so if x here is in A, then there exists one of these finitely many i, where i is one of these finitely many so that the distance. to this xi is smaller than 1.

We would like to say that a is contained in a large ball. Now you just pick one of these xi's. Let's pick the first one. And let's look at the maximum of these-- from the first one to the other one. And let's add 1 to that. You take the maximum of finitely many numbers. So this is a nice finitely many numbers. And you call this here r.

Now you look at the ball of radius r around x1. And I want to show that any x-- so I want to show That for any x in A, then we have that x is in this ball, like that.

So now we know that-- so since x here is, we know that x is contained in the union of this ball. So we know that x is contained in this ball here for some i, where it's one of these little n many i's.

So this means now that the distance-- and so you know that the distance to this little i is at most 1. And so now if you're looking at the distance from x to the first one, then this here is less than the distance from this to xi plus the distance-- this is the triangle inequality again xi. Maybe let me just make very clear that this is the first one to the first one. This is the triangle inequality.

But these distances here are-- this thing here is bounded by $1. It's actually strictly less than 1. So this is less than 1 plus this thing here. And you see, this thing here is, of course, this is how we define r. This is actually less than or equal to r. And so you have this thing here is strictly less than r. And that's what we want wanted to prove.

So this proves that if you take, again, a metric space and you have a compact subset, then it's automatically bounded. What about the second one? Suppose you take a subset. Is it closed? Now let's use the theorem that we just proved. So we need to prove-- so let's prove that this A here is closed.

So now if A is closed-- so to prove that A is closed by the theorem we just proved just a little while ago-- theorem we proved earlier, we need to show that if xn is a sequence in A and xn is converging to some x, we need to show that x is also in A.

Suppose now we want to show that this leads to a contradiction. We want to get a contradiction. So you have this x here. And you're assuming that x is not in A. Yet you have a sequence in A that somehow tends to this-- this is the sequence xn, and it's heading to this element.

Well, we saw earlier that we define something called E here, which is the x, the y in x, so that the distance from x to y is strictly bigger than r. Then we saw that this here is an open set. We show that this is open. This is for each r.

So I'm going to now find a cover of A. I'm going to find an open cover. And the open cover is that for each integer, for each n here, I'm going to look at E 1 over n. So this means that I'm looking at the y so that the distance from y to this x here that I want to show is an A, that this thing here is bigger than 1 over n.

But now I claim that A is contained in the union of these guys because, you see, actually, these guys here, if you have something that is not in any of these, this means that this inequality fails with each n, but this means that-- so if set is in the complement of the union of these guys, actually, then you have that this thing here is less than or equal to 1 over n for all n. You have that. That's a complement of the union.

But if it's less than 1 over n for all n, it means that it must be 0. So this means that this distance here-- so you conclude that this distance is 0. But this means that this set here is actually your x. So the complement of these guys, the complement of this union here is simply just a point x. But you are assuming that x was not in your set A. The complement here, the union of these guys, we have proved now that the union covers everything but x. So in particular, it covers A because we're assuming that little x was not in A.

So this is a nice open cover of A. But A was supposed to be compact. It means that finitely many of these cover. But now this family has this property that you see-- these guys here, if you're looking at E 1 over n, this thing here is contained in E 1 over n plus 1. If this number here is smaller, then it means that this set here is larger.

So you have this-- and so you see that if finitely many-- if A here was contained in finitely many of these guys here, if it was contained in a union of finitely many, then you just take one of these n that is the largest. So this means that A here would be contained in some 1 over for some i because there's one of these numbers-- these ni's, that would be the largest.

But this means now that A-- that there is some lower bound for the distance to x in A. There's no elements that is-- so this means that there exists-- that does not exist, let me say, for all y in A. Then the distance here from y to x is strictly bigger than 1 over this in i. This is just some fixed number. This is obviously po-- you know, right?

But this means that you can't have a sequence in A that is converging to x because they all stay away from x. They stay a certain distance away from x. So this is the desired contradiction. Therefore, there is no sequence that converges to x and the xn on A.

But that's what we assumed in the beginning. We wanted to check. Then whenever you have a sequence in An that is converging, then the limit was also. So that was supposed to have been such a sequence. So that's a desired contradiction.

So this proved that if you take a compact metric space, then if you take a metric space and if you take a compact subset, the compact subset could, of course, be all of x. That's fine. But I want to allow more generally that just to say that just having a subset that is compact, this means that it's bounded and closed.

If you are in Rn-- so you have a theorem, and we'll talk about that next time also, that this theorem-- let's say the converse is the case. But that is not actually-- that does not generalize to metric spaces. So it says theorem here. Well, I mean it does generalize, but not like this. This particular statement does not hold.

So in Rn with the usual metric, say the usual Euclidean metric, if you will. You can just think about R. If A is compact is closed and bounded, then A is compact. And I wanted to talk about an example, but I don't think I have enough time to actually talk about that. Let's talk about this-- return to this next time because we only have four minutes left.

And so, instead, let's prove a theorem here that-- if we take the theorem, if you have a metric space, A here is a subset of x is compact, suppose that C here is a subset of A and C here is [INAUDIBLE]. So this is, of course, in x. So C is also a subset of x. And C is closed. Then the claim is that then C is also compact.

And we have three minutes, and we don't need three minutes because it's actually very quick. And so the thing is-- so proof of this-- so C here is assumed to be closed. So this means that since C is closed, this means that the complement of C is open.

So now let's let O alpha be a cover of C. So O alpha is a cover of C, and now you can add the complement of C to it. So if you're looking at C-- so C here is now, of course, in fact, A here, you can think about A as-- so A is clearly contained in C union with the complement of C.

And so this means that A here, the O alpha cover C-- so this is contained in the union of O alpha union with this last set. These guys are assumed to be open. This here is open because C is closed. So this means that you have now A is contained in-- this is an open cover of A. So this means that finitely many of these-- so this means that A is contained in like O1 union up to some On. And then, quite possibly, you would also need this one here.

But now you see, but C here is contained in here. So this means that C is contained in this. This here was not part of the cover, but it's not needed if you just want to cover C because none of these elements cover anything in C. So this implies that C actually must be contained in O1 union up to On because this here does not cover anything in C. So this proves that C-- there was a finite subcover that covers C.