[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: Last time, we talked about-- one of the things we talked about last time was series. So we'll continue on that today. So series is that you have a sequence. You're starting with a sequence. So a n is a sequence.

And then you form a new sequence. So you form a new sequence. A new sequence from a n as follows. So the new sequence, let's call that Sn, and S1 is just a1, S2 is a1 plus a2, S3 is a1 plus a2 plus a3, and so on. So this is the sum of the first n elements.

And you typically write a Sn. You write Sn as the sum here from-- and then, in this case, I'm starting at 1. But you could start at something else, i equal to 1, to n of ai. And so to say that's the-- so and you write the series. You also write-- so this is slight abuse of notation, that you write this infinite sum, sorry, as the series.

So you think about this as-- you think about this thing here. When you write this thing here, you really think about the sequence Sn. So write this for the sequence Sn. And so the series converge. This means that the sequence Sn converges.

And so the most-- so remember that a sequence converges-- so recall, by the Cauchy convergence theorem-- so recall by the Cauchy convergence theorem that Sn converge if and only if Sn is a Cauchy sequence.

And a Cauchy sequence just means that if you go sufficiently far out and you're looking at this difference here, then this here is small, provided that m and n are larger than some capital N, where this here is chosen depending on epsilon. That's what it means to be a Cauchy sequence.

So in particular, of course, if you take-- so in particular, if you take m to be n plus 1-- sorry. Let me say m to be n minus 1. Doesn't matter.

So if I take it like this, then Sn minus Sm. Well, this here is some of the first n elements, and from that we have to subtract the sum of the first n minus 1 elements. So this here-- just leave this here. Cancel out all of these. And so you're just left with a n.

So this means that if it is converging, then in particular, a n here has to go to 0 as n goes to infinity. Because in fact, this whole difference here, as long as m and n are sufficiently large, has to be as small as you want. And so in a particularly special case where this here is the one prior to that, then you see that the a n will have to go to 0.

So again, if you see a series, the first thing-- you do don't apply any tests. The first thing that you check is that this element here goes to 0. Now what are the most important series? So the most important is the geometric series.

And the reason it's the most important is that it can be used to determine-- it's easy to determine whether it's converging or not. And it can be used to determine using the test whether other series converge. And so the geometric series is that you take a sum.

Typically, here, you start at 0. You don't have to start at 0. But of course, the limit. The limit. If you take a series, then, to say that it's converging, meaning that the sequence Sn converges, and the limit of these here is also. And that's where there's this abuse of notation. It's denoted by the same thing. So this thing here is playing the role of the sequence, but it's also denoting the limit. That's just a standard thing. So in this case, I was starting from 1.

So now your matrix series is where you take some number, c, and then you multiply it by itself. This is some real number, and you multiply it with itself i times, where i runs from 0 to infinity. And when i is zero, this is meant to be 1.

And the reason why I want to do it here at 0 is that it's probably worth remembering that this here, it's easy to deduce. But it's such a standard fact that this here is converging.

So the geometric series converges, provided that the absolute value of c is less than 1. And when it is, this here is the limit. So then you can ask if c here is 1, so if c is actually equal to 1. Not just the absolute value, but if c is actually equal to 1, then it's just 1 times itself a number of times. But that does not go to 0.

So that's clearly diverging. And if it's minus 1 multiply itself by a number of times, then it just alternates between 1 and negative 1. So it doesn't go to 0. So it's divergent here.

And in fact, as long as c here is bigger or equal to-- the absolute value is bigger or equal to 1, because you take this number and multiply it by itself a number of times, if the absolute value of c is bigger than 1, then it actually is even blowing up. It becomes larger and larger in absolute value.

So it's divergent here and it's convergent. So that was kind of easy. And it was easy to determine why we did-- this little trick will determine what this limit was. And this is a trick that goes back, I think, certainly to the ancient Greeks, I mean, this idea. And you can do this with various other things, these things about adding things in different order kind of stuff, or multiplying by something.

So that was the most important. And again, we'll come back to why it is the most important, but it's because it can be used to determine whether other series converge or not. Another classical, but that sits inside a family of series is the harmonic series. So this is an example of a series. It's one series that sits inside a larger family of series.

And so the harmonic series is when you take 1 over n and you're summing here like that. This series here, and we saw that last time, is divergent. And in fact, the larger series that is sitting inside is the-- sits inside a larger family.

And that larger family is where you're summing from 1 to infinity of 1 over n to the alpha. And here, alpha is taken to be bigger or equal to 1.

And so in the case where alpha is equal to 1, that's a harmonic series. These ones diverge. But if alpha-- and so this here is for alpha equal to 1. It's diverged because it's the harmonic series.

And when alpha is strictly bigger than 1, what we just-- last time, we just saw one example of this when alpha was equal to 2, and we saw that that converged. But in fact, you can prove that for any alpha strictly bigger than 1, it does converge. And the typical thing to use there is-- and we'll talk about that at some later lecture is what's called the integral test.

Now there is-- when you have a series, then most, but not all-- most tests, but again, not all, determine really whether the series is what's called absolutely convergent. So what is absolutely convergent? We talked about that last time. But let me remind you.

Absolute convergence. So if you take a series here, then here is absolutely convergent. Is absolutely convergent. This just means-- it means that the series where you're slapping on absolute value, so this thing here, means that this series here is convergent.

And so it's easy to see. So easy to see. And we saw this last time. And it was just from the Cauchy convergence theorem. It is easy to see that absolute convergence implies convergence, but not the other way around.

So Cauchy convergence theorem gives you this implication. Why it's not the case that you have the other way around? And that is because what's called the alternating series. So the alternating series. Alternating.

And there's something called the alternating series test. But let me just explain to you in words why the alternating series converge. So the alternating harmonic series.

So if you take this series here minus 1 to the n times-- over n and you sum here from 0 to-- sorry, not 0, because you're dividing by 0, so from 1 to infinity, this here is called the alternating series. Alternating harmonic series. If you were slapping an absolute value sign on it, this means that a n here is minus 1 to the power n over n.

So if you take the absolute value sign of a n, then this here is just 1. So this is 1 over n. So this here is the corresponding series when you're slapping absolute value. And this is the harmonic series. And this is divergent. That was what we just discussed.

Now why is this series here convergent? We'll probably come back to that in a later lecture in a more general setting, but let me just explain this. And so the idea is that before you start summing, imagine that you have here-- before you start your sum, then you are in a neutral position, which is 1. 0, sorry.

And then you do the first thing. You're looking at S1. But S1 will bring you down, because S1 is just equal to minus 1 to the power 1. But that's minus 1. So it's minus 1 over n. This is the first element.

So that's going to bring you down. Let me just exaggerate the picture and bring you down here to minus 1 over n. Minus 1 over-- n is equal to 1. So this is minus 1. This is how the series starts.

And so the next element is S2. But S2 is minus 1. That was the first element. And then plus and then minus 1 squared over 2. So this is-- and you would be tempted, but it's not-- don't add them at this stage. But just looking at it like this.

So the next-- so this here is S1. Let me just write it down here. This is S1. When you're looking at S2, then you're coming back towards 0. But you don't go as far up as you went down, because it's 1 over 2. Right. So this here is S2.

And then you see the next one is-- the next one. If you start here, then you are supposed to-- so for S3, it's really S2 plus minus 1 to the power 3 over 3. So this is S2 minus, because minus 1 to the power 3 is a negative sign like that.

So the next one here-- so you're starting here before you have done any summation. You're starting at 0. You're brought down here. You come up here. But then in S3, you're going down, but you don't go as far down as the previous one. So you're going down here.

And so you're really continuing this process. And you see that this interval gets smaller and smaller, and it just goes down, and then it come up. But it doesn't go as far up as you were before, and then so on and so forth. And so you could make this more precise. It's quite easy. And this shows you that this series is converging. And in fact, this is the basis for this alternating series test. And it works as long as you have alternating-- where the sign of the a n's are alternating and the absolute value goes to 0. Then this here, that's your alternating series. So I'll probably come back to that at some later stage.

But this just shows you that if you take a series, then-- So the absolute convergence means that if you slap absolute value sign, then that new series also converges. Absolute convergence implies convergence, but not the other way around.

So now the next thing is that-- so the next thing is that if you take a series-- suppose you take a series and suppose that you knew that-- so suppose you have a series like this and you knew that a n here-- all of the a n's were non-negative for all n.

Then convergence here. Convergence. Just implied, right? So convergence is-- because the Sn is now the sum here of these guys here. That's Sn. And you see that this is, of course, less than or equal to-- because all of these guys are non-negative, it's less than or equal to where you summing. Not just up to n, but one more, because this is non-negative. And so this is Sn plus 1.

So you see that if the elements are non-negative, then the sequence Sn is-- so Sn is monotone non-decreasing sequence. And so to say-- by the monotone convergence theorem, to say that this sequence, Sn, is convergent, is equivalent to saying that. So Sn converges if and only if Sn is bounded from above.

OK. So now when you have a series and you have already checked that a n goes to 0, and you are then asked to determine whether the series is convergent or not, then there are these number of tests. And so there's a bunch of tests. So tests for convergence.

And so there's a bunch of them. And let me talk about-- in this lecture, I'll just talk about the comparison test. And I mentioned them very briefly last time, but I'll talk a little bit more about them. And then there's the ratio test.

And then there's the root test. So I'll talk about those three. But there are other tests. So root test. So now let's start with the comparison test. And there are two versions of this. One version is the following. Let's suppose that you have-- so comparison test version one.

And so the first version of the comparison test is the following. Imagine that you have two series here, a n, and again, we talked about this last time. But let me just say a few more words, that you have this series and you have that series.

And I'm assuming here that a n and the bn are non-negative numbers. And in fact, I'm assuming that bn is larger or equal to a n. In the comparison test, this version, the simple version of the comparison test is this here. If the series based on the bn is convergent, then the series of the a n's is also convergent.

And so this version of the comparison test is used both in this way, but it's also used sometimes where you know that this series is divergent. And from that, you conclude that this one is just a negation of this statement.

So let's see. So the proof here of that is actually easy. So why is this the case? Well, the thing is that you know by the Cauchy convergence theorem.

And we know that to determine whether-- all we need to check to determine whether this sequence here is convergent-- all we need to figure out is if it is a Cauchy sequence. So this means that you want to say that-- so suppose you're looking at the sequence defined from a. And so this is just where I'm now summing the first n elements in a. And I just want to-- so I'm using the superscript a just to denote that it's coming from the sequence a n.

So if you're looking at this, and so likewise, I look at this thing here. Now, of course, you have that suppose you're looking at Sna minus Sma. You want to show that this here-- you want to show that this here is a Cauchy sequence. So you're looking at this difference here.

But this thing here is just this here difference. I'm assuming that m is less than n. Strictly less than m. n, sorry. This just means that this difference here is just where you're coming from m plus 1. So this is just the a m plus 1 up to a n. This is just this one. Yeah?

AUDIENCE: When we're proving convergence, there's now-- just to give some motivation, is the reason that we're using Cauchy sequences?

TOBIAS COLDING: At this stage, you could either use Cauchy sequence, that's what I started. You could also have used the monotone convergence theorem. Maybe that would be--

AUDIENCE: OK. Because I was just going to say, back to the definition of convergence that we initially had where we had the absolute value of a of n minus some--

TOBIAS COLDING: Yeah, that's right.

AUDIENCE: Is it just like-- is it--

TOBIAS COLDING: So if you went all the way back to the definition, then you would have to determine what the limit was.

AUDIENCE: Right, which we don't want.

TOBIAS COLDING: And we don't want because we don't know. But just because we may know what the limit of b-- of the sequence based on b is. But this doesn't mean that you can figure out what the limit of the a--

AUDIENCE: So does that motivate why the sequences are more useful in general?

TOBIAS COLDING: Yeah, that's right. That's right. That's right. Exactly, that's right. That's right. So the advantage with Cauchy sequence is that you don't have to figure out what the limit is. And it's often-- I mean, like in this case, you have no idea.

I mean, it's not in terms of anything from the b's. And I should-- but let me just continue doing it like this. But just now when you asked, I should probably have done it using the monotone convergence theorem. But you can do that as an exercise. But it doesn't matter.

So the difference here is-- this is what the difference is. The absolute value signs are really irrelevant, because these are all non-negative numbers. So I don't have to have the absolute value sign.

And now this thing here is, of course, less than. But this here is less than the corresponding b's. And this is less than the corresponding b here like that. But this here is exactly Sn, the one based on bSm. Based on b. And this here, as long as n and m are sufficiently large, we can make this as small as we want. So therefore, this here was the smallest one.

You could also have done it-- maybe let's just say that you could also have done it using the monotone convergence, because now, of course, you have that. Again, the same of thing. Say that if you take the Sn based on the a's. So that's a1 up to a n sum of these guys. But this individually is less than the corresponding b's. And this here is just the sequence Sn based on the b's.

But these here converge. This is equivalent to saying that they are bounded. They are less than some c-- some real number c. And this here is for all a. For all n, sorry. But this means, of course, that those things here are bounded for all n. And this here is also a monotone sequence-- monotone increasing sequence that is not bounded from above. So therefore, it converges.

So you can do it either of those two ways. So that's really easy to check. Let's see an example where we are using this test here. And so this could be that you're looking at something like-- you're looking at-- so here's an example.

You're looking at a series that is true to the minus n over n, n equal to 1 to infinity. And you want to show-- you're being asked, is this series converging? So this is typically how it works.

And so, of course, in this case, you should think about that this here is a sequence a n. And so you have to find the bn. But in this case, we just say-- so we're setting a. So answer set a n equal to 2 to minus n over 2 and bn equal to 2 to the minus n.

Then you have that a n bigger or equal to 0 is smaller than bn. And the sequence here based on the bn is a convergent geometric series. It's convergent between-- because c here, in this case, is 1 over 2.

It's 1 over 2. So it's just 1 over 2. 1.5 is the same as 2 to the minus 1. And so this is convergent.

So this is just so-- so typically, this could be used to-- that you're given a sequence and you're being asked, is convergent or not? And so you have to find the sequence and you're being asked whether it's convergent or not. And you have to find the comparison series.

And then, in this case, say that this thing here is convergent. And so therefore, the sequence a n that was squeezed between that and 0 was also convergent. So this was the first version of the comparison test. Here is another more fancy version of comparing, where you're comparing two series.

In fact, it's pretty much like that, except for this stuff we talked about. I mean, typically the tests are where you're comparing it with some other series. So here is version two. So comparison version two.

And so you have, again, two series, a n. And again, it doesn't matter-- what the sum is matters where you're starting, but whether or not it converges doesn't matter where you start. So suppose that you have this series and you have this series here.

And suppose that you know that the none of the bn's are 0. And you know that the a n divided by bn, that this ratio here, as n goes to infinity, converges to some L. And L here is-- L here is not 0 and is not infinity. I mean that it really is a limit, like that.

So if that's the case, then the series-- then let's-- then let's have these as non-negative numbers. Then the series a n converges if and only if the series bn converges.

And so an example of this-- so a typical example of this is where you're looking at the series 1 over n squared minus 1. Suppose you're looking at this. And now, just to make it so that-- make sure that it makes sense that you're not dividing by 0, you're looking at this thing here.

Suppose that this is the series that you're given. Well, this here is-- so you wouldn't use the first comparison, although you could have used the first. But then it would be a little bit more complicated. Not super complicated, but a little bit more complicated, because this thing here this is-- so the first thing you observe is that this, of course, goes to 0.

This goes to 0. But the second thing is that it's very close to a series that you know something about. It's very close to this series here. But again, it doesn't matter where I'm starting when you're asking whether it's converging or not. So it's very close to this series. But of course, this thing here-- this thing here is a smaller number. So this is actually bigger than 1 over n squared.

This here is a convergent series. So this does not help you. This is not great for using the first test. You could have done it. You could still use the first test. But then you have to build a slightly more clever.

So instead of that you can use this second test. And so in the second test you, set a n equal to 1 over n squared minus 1. And you set bn equal to 1 over n squared. And then you have that a n divided by bn. But this is just 1 over n squared minus 1 over n squared. Or 1 over n squared, sorry.

And so this here is the same as n squared minus 1, like that. And now you can divide by n squared in both numerator and denominator. So you're getting 1 minus 1 over n squared. And now you just use the usual algebraic rules for limits.

This here converts to 0. So it means that the denominator converts to 1. So this means that this ratio here converts to 1. And so you see that the ratios are converging to something that is neither zero nor infinity. And this means that this series here is convergent if and only if this convergent. You already know that this is convergent. So we conclude that this is convergent. So that's how it's typically used.

So now the next test I want to talk about is the ratio test. And so this is-- I mean, the comparison tests are typically almost the easiest tests. But this one here is extremely useful.

And so the next two tests, the ratio test-- both the ratio test and root test, are based on comparison with the geometric series. And so this is explaining why the geometric series is so crucial.

So what is the ratio test? Well, the ratio test is simply the test that if you're looking at-- so if you have a series. You don't have to look at any other series. Like in the comparison test, you had to find another series. But in the ratio test-- so this is the ratio test.

So you're looking at a series like this. And the ratio test is that you're looking at a n plus 1 over n. So you are looking at the ratio between two consecutive elements of the a n's. So you're looking at this and you're looking at the absolute value of this. And you're asking if the limit here-- as n goes to infinity, suppose that the limit exists. And let's say that the limit is a. So it's asking you if this-- somehow the test here-- this form of the ratio test, it requires that the ratio here has a limit.

So the ratio test then says that if a here is-- if a here is-- sorry. If a here is less than 1, then the series is-- so let me write it like this. So a less than 1, this here means that the series is convergent.

Two, if this a here is bigger than 1, this implies that the series is divergent. And three is that if this a here is 1, then the test is inconclusive.

But it's important to remember that it is possible that you don't get anything out of the test if you are in the unlucky case where this limit is 1. And I should also emphasize that it is crucial to have a limit. So the test here, in this form, it's crucial to have a limit. And we'll talk a little bit and we'll come back to that. We'll come back to this.

So now let me try to explain why you have-- well, in conclusion, I don't really need to explain anything. But why in the case-- so proof of case one. So this means that I have to show-- so I need to show that if this limit here-- if this limit of a n plus 1 over a n, if this here is less than 1, then the series is convergent.

And I'll just sketch the proof, because we're going to do-- yeah. I will just sketch the proof. We;ll come back, in a way, to the proof, because it's very close to the same proof that is used in the case of the root test.

So that's what I need. So I have that there exists-- so given a0 less than 1. So given a0 with-- a0 bigger than this a and less than 1. So if you pick an a0 that is strictly bigger than a and still less than 1, then we know that there exists an n such that if little n is bigger than this capital N, then a n plus 1 over a n-- and really, I should have absolute value here on this.

Then we have that this thing here is less than-- it's less than a0. Because if you're very far out, then this here is-- this ratio is going to be very close to this. But this is strictly less than that. So we have this.

So this means that-- maybe I shouldn't call it a0 because there's too many a's now. Let's call it something else. Let's call it alpha to make it just look much more distinct.

So this means that-- so you see that if-- so in other words, If n here is bigger than this capital N then you have-- just multiplying over, you have that a n plus 1 is less than alpha times a n, like that. In the ratio test, I should have said that. But obviously, you are assuming that these a n's are not 0. So you can divide by them.

So you have this. If you are far out, you have this, that the next one is a fraction of the previous one. But this means, of course, that if you take-- if you're looking at n is bigger or equal to this capital N, well, you have that a capital N plus 1.

If you're using this a capital N as little n, then you have that this thing here, this is now the-- this here is now-- sorry. If you're using-- if you first pick n equal to capital N, then what does this inequality tell you?

It tells you that a capital N plus 1 is less than or equal to alpha a capital N. Now apply this inequality here for n -- plus-- capital n plus 1. So then you have that a capital N plus true absolute value is less than or equal to alpha a n plus 1 like that.

But you already proven that this thing here-- forget about the alpha here. But you've proven this inequality. So filling in that inequality, I have this alpha here. But then I have another alpha that is this and then like this.

And so now, you can continue this way. So this here is like alpha squared times a capital N. And so now you see that you can continue. So if you're looking at-- so if you continue like this, then you see that a capital N plus some little n is less than or equal to alpha to the power m times a to the absolute value of a n. This here, the previous line, was just where m was equal to true. But you really get it for all integers.

And so this says that at least the tail of this sequence here, the tail of the a n sequence, this is just some fixed constant. This here is a convergent geometric series. So the tail is bounded by a convergent geometric series. So you can just apply the first comparison test to prove that the sequence-- the series a n is actually convergent. So the ratio test, again, was to compare it, use the first comparison test where the comparison was the geometric series.

And so of course, if the-- so this proves one, two here. But two is kind of really the same. Because if this alpha here-- so to show the second part. Again, the third one, to prove it's inconclusive, there's nothing to prove, because it leaves open any possibility.

But you want to show that if this limit here-- so if this limit-- so proof of two. So you want-- so we're assuming that a here is bigger than 1. But this means that there exists-- so if you take alpha-- so this time, choose alpha such that alpha is bigger than 1, but it's smaller than a.

But then you have that for-- since this ratio is converging for little n bigger than-- there exists some capital N, so that if this little n is bigger than capital N, then you have that a n plus 1 over a n, that this thing here, if it's far enough out, it's going to converge here. But at least far enough out, it's going to be bigger than alpha.

Multiplying over again means that a n plus 1 is bigger than alpha times a n. This here is now bigger than 1. And so in the same way as before, you see that if you're far enough out, then the next element is some multiple, some big number, at least bigger than 1, times the previous one.

But this means that they're getting larger and larger. From some fixed stage on and outwards, the a n's are getting larger and larger. So they don't converge to 0 as they were supposed to. So this is proving true. And three, again, there's nothing to prove, because it leaves open both possibilities.

AUDIENCE: I have a question about the first group.

TOBIAS COLDING: Yeah.

AUDIENCE: Does that assume that the series is monotonically decreasing?

TOBIAS COLDING: Sorry?

AUDIENCE: I guess just looking at it from a high level, we're saying that a of n plus 1 over a of n--

TOBIAS COLDING: Wait, wait. So you're asking-- what are you asking?

AUDIENCE: So for part one, when we say that a is less than 1, that implies that it converges, is that only going to work for monotonically decreasing?

TOBIAS COLDING: No, no, no. So there's no assumption here on whether the sequence is right. I mean, so 1-- yeah. So 1 implies that the absolute value is going down, as you say.

But I could alternate signs and things like that. Yeah. But it is-- yeah. And so just because you're comparing it with-- you're going to compare it with a convergent geometric series. In a convergent geometric series, the next element is some fixed fraction of the previous one. Yeah.

So you're right that in order for the ratio test to give you convergence, then it's going to require that sufficiently far out, the absolute value of the a n element has gone down.

And this, in a way, addresses that point you're making there. So the other test that is also based on comparing it with the geometric series. But this does not compare consecutive elements. So it's leaving the possibility that the next element could be larger than the previous.

And so this is the root test. But of course, it cannot be that-- you cannot keep having the next element larger than the previous. But you could have-- there could be somewhere where the next one actually went up. But then presumably, after that, they're going down, at least for a long time. And then maybe there's another one that jumps up, but not very much, et cetera. And the root test is OK with that, whereas that test wasn't OK with that.

So the root test. So you have a series, again, like this. And the root test is that you're looking at-- you're taking the absolute value of this and then you're taking the n-th root. So this means-- this is the same as saying a n absolute value, if you want to write it like this, 1 over n.

So it's dealing with this. And the root test is saying that suppose that when you form this here, that this has a limit. Suppose that this n-th root-- suppose that this here has a limit. And let's call this limit c.

Suppose that you have this. Then it has the same three parts here as in the previous. And so this is-- so there's three possibilities. Where c here is less than 1, then this implies that the series is convergent.

Two, c is bigger than 1 implies that the series is divergent. And again, the case where c here is equal to 1, in conclusion. Test is in conclusion. And again, we're going to compare with the geometric series. So let's see how the proof works.

OK. So now let's prove one. Proof of one. So again, we are in a situation where we have that the n-th root of the absolute value here as n goes to infinity is equal to some c, and we're assuming that the c is less than 1.

So now I'm again choosing something that is between c and 1. So choose c-- let's call it c0 with a-- c0 less than 1. And now we know that-- because we have this limit, we know that there exists capital N such that if little n is bigger than capital N, then the n-th root of this absolute value sign is actually less than or equal to c0.

This is just because c0 was strictly bigger than c. So we have that. So now we just check the n-th power on both sides. So this implies by taking the n-th power, we get that absolute value of a n is less or equal to c0 to the power n.

But that's already great, because this means that you have compared-- at least from a certain stage and an outward, you have compared the a n with-- if you're thinking that this thing here is like the b n, then you have compared these guys here with this series here based on these, at least from this capital N and outwards.

And this is a geometric series because it's just some number strictly smaller than 1 multiplied by itself n times. So again, by the first comparison test-- so by the first comparison test, the a n is convergent.

And note here that what you're proving is-- you're actually proving that this series is convergent. Because you're comparing this series here with the geometric series. But this here is just-- this just means-- sorry, means that the original series here was absolutely convergent. But absolute convergence implies convergence. So you're actually proving absolute convergence.

Now let's try to-- again, there's three parts to it. But the last one leaves open any possibility, so there's nothing to prove. So let's just prove the second part here. And that just goes similarly.

And again, it fails dramatically to be convergent, because the a n, as you will see, does not actually goes to 0. So proof of two.

So we have that this-- so we're assuming that this thing here, this should converge to some c here, which is now bigger than 1. So this means that there exists-- so this means that there exists capital N such that if little n is bigger than this capital N-- sorry.

So we have this thing here. And what I should before I do this, I should choose my-- just like before, I should choose my c0 such that c0 is strictly bigger than 1, but yet is strictly smaller than c.

So you choose this. And now using this, we have-- because we have that this limit here, it converts to c. So it converts to something that is strictly above this c0. This means that there exists capital N such that if little n is bigger than capital N, then the n-th root here is bigger than c0.

But now, again, taking the n-th power on both sides, then you have that a n is bigger than-- strictly bigger than, in fact, c0 to the n-th power. But you see these guys here? c0 is strictly bigger than 1. So this here actually even goes to infinity. So this here does not go to 0.

So this proves, again, that it's divergent. So this proved true. Now the next thing-- the next thing we want to talk about is power series.

So remember that-- so the first example of a power series is something we already looked at, more or less. So suppose you take-- suppose that you're looking at the series, some number, I'll call it x now instead of c, to the power n, and n equal to 0 to infinity. Yeah?

AUDIENCE: [INAUDIBLE]

TOBIAS COLDING: So what we proved-- so in the root test here, we proved that the series is absolutely convergent. Here we proved it's divergent. And what we actually proved is we proved that a n-- in fact, this here a n that is here allows me to write that it goes to infinity like this.

They were supposed to go to 0, but we actually write-- if the series was converging, the most elementary thing is, the most basic thing is that the a n has to go to 0. Otherwise, it's not going to converge. But in fact, not only do they not go to 0, they actually, in absolute value, go to infinity. So it's very, very far from being convergent.

So suppose we looked at the geometric series. If you think about x just representing any number, you can look at this series here. This is the geometric series. And we know that it's convergent. So we know that this has convergence.

It's convergent if the absolute value is less than 1. And in that case, the sum here is just this familiar function. But in general, you may have a sequence. So in general, if a n is a sequence, then you can form a power series.

And the power series is like that, where this is the coefficient. The a n is now the coefficient to the x to the power n. So if you will, this is like an infinite polynomial, a polynomial of infinite degree because the sum of all these polynomials.

And this is called a power series. So this is a power series. So this is a power series. And the question is then-- the question, which turned out to be quite subtle, is-- so a lot of the basic functions can be represented by power series. This here was one example.

Another example is-- so another example of-- if you're looking at the exponential function-- and we'll come back and discuss this more. But the exponential function has a nice power series representation. And this is the exponential function, the power series representation of the exponential function.

And also the sine and cosine. So cosine x, this also has a power series representation. And the power series is minus 1 to the power n x to the power 2n divided by 2n factorial, where you're summing from 0 to infinity.

And sine has a similar one. So sine x has a similar one where you're summing from 0 to infinity. And then it's minus 1 to power n x to the power 2n plus 1 over 2n plus 1 factorial.

And lots of functions have a power series representation. But it's not every function. If I write down a function, then-- so in fact, it's easy to write a function that doesn't have. So if you have-- but I'm not going to prove to you that it doesn't. So if I take the function that starts off being dead zero and now it's becoming interesting. So this function here, this one does not have a power series. Even locally does not have a power series.

And the thing is that a power series is an example of something that's called an analytic function. And this here is not an analytic function. So that's why I know that it definitely doesn't have a power series. But anyway.

So the question is if you take a power series-- so suppose you take a sequence, a n, and you write it down. Suppose you take a sequence a n and then you write down a power series based on it. Then the first question that comes to mind is, when does this-- for which x does this series converge? So for which x does the series converge?

So this is like the most basic thing. And let's look at some examples. Of course, if you look at the-- so the first example is what we already talked about. Suppose I'm looking at this thing here.

Well, this here is x. If you think about x as being fixed, this is a geometric series. And we already know-- so this is for each fixed x, this is the geometric series.

So we already know that this thing here converges. So converges as long as the absolute value sign-- absolute value of x is less than 1. And we also know that n diverges otherwise, because we've already determined that.

Now you see-- the next example I want to look at is the one power series that represents-- but we haven't proven that yet. But we'll talk about that next time. It's the exponential function. If I'm looking at this power series here, then I have-- then I'm interested in for what x-- for what x does this converge?

And so it's natural here to try to apply one of the tests. So you just think about x as being fixed and you just apply one of the tests. And the natural one is either the root test or the ratio test. Let's do the ratio test.

So x is fixed, and I'm not interested in x equal to 0. I'm assuming that x is not equal to 0, because when x is 0, this here is just-- it's only for-- x to the power of 0 is by definition one. And all the other things, all the other terms, are 0.

So this here is-- it's definitely converging when x is equal to 0. So let's not look at that. And then-- because if you don't look at that, you can look at-- you can think about a n. You can now think about, sorry, as the series where bn is x to the power n over n factorial.

If you think about this thing here, and you think about the series as now bn, then the ratio test tells you that if you're looking at the ratio of bn plus 1 over bn-- so if you're looking at bn plus 1 over bn, you take the absolute value of this here.

Well, this is just x to the power n plus 1 over n plus 1 factorial. And then you have to divide it by x to the power n over n factorial. But if you do that, this here flips upside down, and you're getting x to the power n plus 1 times n factorial. And then you're getting x to the n times n plus 1 factorial. So this is what it's equal to.

And now, of course, here, there are x times itself n plus 1 times. The n of those is canceling out with this. So this is x. And here, you see here the n plus 1 factorial. You can write as n plus 1 and then times n times n minus 1 cetera. But that's just n factorial.

So you can write it like that. And so you see that this here cancels that. So you're left with n plus 1. Remember, if x is fixed as n goes to infinity, this here-- for each fixed x, this just goes to 0.

So now you have, from the ratio test, if that limit-- if it has a limit-- it does have a limit. If that limit is strictly less than 1, but it's actually 0. So it's definitely fallen under 1. So it's convergent. So the ratio test tells you-- so by the ratio test, this is convergent.

And so we'll finish here. There's a little bit more. I posted already my lecture notes. There's a little bit more. And I'll probably talk about that next time.

I'll just start with just talking about the stuff I didn't get to. The stuff I didn't get to was-- and we touched upon that earlier-- is lim sup and lim inf. And the reason why we're covering it here is that what if the ratio-- what if these things-- what if the n-th roots-- so you're asking for either the ratio or the n-th root of something to have a limit.

But what if it doesn't have a limit? And so it's dealing with that case. OK. All right. Any questions?

AUDIENCE: That means for all x, right, you're saying?

TOBIAS COLDING: This is for all x. But think about x as being fixed. Think about x. You're just asking for one particular x, but it holds for all x. But don't think about x also as being variable right. Because you think about x as being fixed, and then you let n go to infinity.

AUDIENCE: Right. And then after, you generalize.

TOBIAS COLDING: Yeah, yeah. That's right. And so you have for all x, this makes sense. Then you can ask, how does it vary in x? But that's a different story.