[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: OK, so assume that again that we have-- so this is a sequence. Then from a sequence, we can form another sequence. So from a-n we can form another sequence, at least. If a-n is bounded from above. So we can form another sequence, and that's what we take.

B-n is going to be the soup of the set of a, starting at n. So this here is the sequence b-n, OK? But again, implicitly here we are assuming. So assume that there exists some c, so that for all n a-n is less than c. In other words, the c are bounded from above, right?

So if the sequence a-n is bounded from above, then this here gives us a real number, and we see that if you're looking at b-n plus 1, this is now sup of a-n plus 1, a-n plus 2. And this here is sup over a smaller set, so this is less than the sup where we take sup over a larger set. And that here is b-n.

So we have that if you take any sequence, and let's say that that sequence is bounded from above, then you can form another sequence b-n, and b-n here is monotone decreasing.

If the sequence b-n is bounded from below, then the b-n here in the sequence here converge, and it converges to the infimum, right? This is this with monotone converging. Sequences, it's converging to the infimum of the b-n like that. OK.

So likewise, although we won't really use this here, but likewise, again, if a-n is any sequence, any sequence, this time I want it to be bounded from below. Then you can find a sequence. From a-n, you can form a sequence, c-n, and c-n is just where you're taking the infimum, starting at a-n.

And you see that if you're looking at c-n plus 1, then you take the infimum over a smaller set. So this is a smaller set. And so this here is bigger than where you're taking the infimum from a-n, and that's c-n. So the sequence c-n is therefore monotone increasing. And if it is bounded-- so if c-n Is bounded from above, then the sequence c-n converge to some c, and that c is just a sup of the c;s, right?

And we talked a little bit about this earlier, but now we will use it in just a second. The thing is, though, that in the first case here, we call b here, B is written as the limsup of the a-n. And in the second case, it's called the liminf of the a-n. OK.

So why is this relevant here for us? So what we were looking at last time-- and I'm only going to use the limsup. So last time, we had a sequence. So a-n was a sequence, and we defined what we call a power series. And again, power series, think about that x as being just some fixed number, and you have the sequence a-n. X is some fixed number, and then you form the sequence a-n x-n, right?

And so for each x fixed, this gives you some number. Maybe, if it's converged. I mean, you can ask for each x. You can ask, does this sequence converge or not? And this here is called a power series. So you're summing from 0 to whatever. Or you're starting from one, or whatever you're starting with up to infinity. So this is a power series.

And so now for power series, the question is, how serious is the question is, for which, for which x does it converge? So that's what we're interested in,.

So just let me remind you, because we're going to use that anyway in a minute. So example from last time. The example we looked at last time was we're looking at x-n. And we looked at some other examples, too, but this was the one we're focused on. So this here is a power series. The a-n in this case here are 1 over n factorial.

And for each fixed x, so for x fixed we looked at whether or not this sequence converges. And so we think about this thing here. We thought this here. We thought about it as just a sequence like this, sum of a sequence. So this is as a series like this, where the d-n here is now x to the n over n factorial.

And so in order to determine for which x it converges-- so just think about x as being fixed. We use the ratio test. So we use the ratio test, and the ratio test said that if you're looking at d-n plus 1 over d-n, this here is where d-n is just x to the n plus 1, over n plus 1 factorial, divided by x to the n over n factorial. And that's the same.

So when you divide by a fraction, you just switch. So this is the same as x to the n plus 1, divided by n plus 1 factorial, times n factorial, times x to the n. And you just think about this, the top here, as x to the n times x, times n factorial. And the denominator you think about as n plus 1 times n factorial, times x to the n. And you see that this here and this here cancel with those two. And so you get x over n plus 1. And this here converges to 0, so therefore the power series is convergent for all x.

So that was a particular example of a power series, and it worked well with the ratio test. But the word test actually is better. And this was related to a point somebody raised last time. And that point was for the ratio test, it was implicitly somehow in the ratio test, was something about that from some stage on, the next element is smaller than the previous one. And you don't really need to have that. I mean, you certainly don't need it for a series to converge, but it's not even necessary if you apply the right test.

And so this brings us to what's called the radius of convergence. So this is a crucial concept, and by this crucial concept, because you see a priori what you could imagine-- so suppose you just write down a power series like this. Maybe it's convergent for when x is equal to 5, it's divergent when x is equal to 10, and maybe it's converging again if x is equal to 20. That's not actually going to be the case.

So there's something called radius of convergence. And the thing is that what you get out of this radius of convergence is that a power series like this is convergent on some interval, and then outside this interval is divergent. So it's not like you have some that is convergent here, it's divergent there, and then it's converging again. That's not what's going to happen. OK?

Part of this we already saw in the different tests that at the boundary of this interval, the test is inconclusive, so you have to check the boundaries separately. And so the idea is that you check-- so you have a power series like this, and you think about-- again, you're thinking about d-n as x to the n. X, again, is thought of being fixed, and now you apply the root test.

So apply the root test, but I'm going to apply it in a better form than what we did last time. Maybe I'll just state first the better form of the root test. Here's the real version of the root test. So suppose you take a series. Suppose you have some series here. Let's just call it d. Suppose you take a series like this.

So now the root test, you took d-n, you took the absolute value, and you took the n-th root. When I looked at the root test last time, we looked at whether this is convergent or not. So we said if it was converging to something, right? but you could form this thing here. So you could form from this here. You could form this. This is now a sequence, and you could look at limsup of this.

These here are all non-negative numbers, and this limsup here is now going to be either infinite or it's going to be some real non-negative number. This here is going to be infinite. Usually, we don't talk about that it's converging if it's infinite or it's some number. Let's call it d, I guess. OK?

And the thing is that if you said r equal to 1 over d-- sorry. This is just a root test, the adult version of the root test, because this applies to all sequences. For any sequence, you have this. For any sequence d-n, if you're looking at this series, you can look at this, and it's either infinity or d.

And then if d here is less than 1, then the series is convergent. If d is bigger than 1, the series is divergent. And if d is equal to 1, it's inconclusive.

The difference between this and what we talked about last time was that last time we assumed that this thing here converged. Here, I'm not assuming it, but then I'm looking at the limsup instead. So this does give us a number, and it's just a matter of whether this limsup is smaller than 1. And then it's convergent or bigger than 1, then it's divergent.

Now I'll come back to the proof, but the proof is really kind of the same as the one we gave. The one we gave, really, if you mimic what we did in the special case last time, then the proof would still apply to this case here and give this result. But let's take this more fancy version of the root test. Let's take that for granted.

So now I have a power series like this. Just for clarity, I define d-n to be just a-n, times x-n. And I'm interested in is for what x, for what x does the power series converge?

So you are looking at limsup of the n-th root of d-n is a-n times x-n, and you take the absolute value of that. That's what you're looking at, and you're asking is this smaller than 1 or bigger than 1?

But this thing here, right, this is the same as limsup, right? If you're looking at this thing here-- let's do a little side calculation-- if you're looking at the n-th root of a-n times x-n, this thing here is the same as the n-th root of this product here. Absolute value is the same as this times x absolute value to n. And when you take the n-th root, you may as well take the n-th root of each of those two factors individually. So this is what you get.

But this thing here, you see, this thing here, the n-th root-- this is whatever it is. Can't do much about that. But this thing here, that's just the norm of x, and the norm of x has the property that when you raise it to the n-th power, you're getting this. So this is the n. So this means that this thing here is the same as the norm of x limsup of the n-th root of a-n. Often you do it like this. You don't have to do this. It's usually clear anyway, but often you write it like that.

So now what you do is you are asking when is this number here smaller than 1? When is it bigger than 1? You see, now you define something that's called the radius of convergence. And so this is the following. So the radius of convergence-- So this is just radius of convergence. The radius of convergence is just a number, and that number is radius of convergence. And we have to just interpret it a little bit. So it's 1 over limsup of the n-th root.

This thing here could, of course, be infinite. If this is infinite, then the radius, by definition, this here would be 0. Where this here is equal to 0 if this limsup is infinite. And otherwise, it's always a-n. Otherwise, there's one other case you have to define, that if it's infinite, if the limsup-- sorry, this should be absolute value if this limsup is zero. otherwise.

Otherwise, there's nothing. It's, well, that's the only thing you have to comment, because otherwise, it's always a non-negative number. But it could be 0. It could be infinity. But if it's any other number, then it is just what you think it should be. It's 1 over that number.

So r here is the radius of convergence, OK? And now the thing is that now we have the following symbol, picture. So now we have that. So you have this power series. Again, it doesn't matter what you start with. Some matters what you start with, but not the convergence.

So you have this power series. You are forming r, this is the radius of convergence. And then the claim is-- and again, this could be 0. It could be infinity. But often, it's just a real number, a non-negative real number, positive real number.

And so, it's saying that if you take the interval here around 0, in the interval from minus r to r, then inside this interval here, inside here when x is in here, then the power series is convergent. And when x is here or here, power series is divergent.

And then when x is here or here at those two points, then it's unclear and further test is needed. Further test is needed. Tests are needed. OK. Yeah?

AUDIENCE: Starting from the last step on the left to get to this equation for r, are you just essentially saying that less than or equal to 1 or less than or equal to 1? But then we only care about one x less than 1 because we need that.

TOBIAS COLDING: Yeah, so that's right. So let's come to the proof. Yeah. So this is what you're talking about. So this is the radius of convergence. And so as you say, all we need to do is-- we know from the more fancy version of the ratio test, that is, in terms of limsup, we know that--

So proof here. Power series is convergent by the ratio test. Not ratio test, sorry. The root test, by the root test. So again, the version, the limsup version-- not the one I talked about last time, but this one we just talked about-- by the limsup version of the root test.

Then the power series is convergent if this thing here-- so this means that this thing here-- so if this thing here, limsup of an n-th root. If this thing here is less than 1, it's divergent. If this thing here is bigger than 1, and it's inconclusive if this thing here is equal to 1 here. It's inconclusive.

But now you see that this here just means that the norm of x should be smaller than 1 over that. So this here is the same as saying that the norm of x is smaller than r, the radius of convergence. This is equivalent to saying that the norm of x is bigger than r. And this here is equivalent to saying that the norm of x is equal to r.

So that's the radius of convergence. So this is, like, a priori. This is not at all clear that it couldn't be-- that a power series couldn't be convergent for some sort of discrete set of x. But that's not the case. It's convergent on some interval. It's unclear at the end of that interval, but it is convergence on the interval, and divergent outside that interval.

Now that interval could be just 0 or it could be all over. But it is also symmetric around zero.

AUDIENCE: But when we bring that over, isn't it 1 over?

TOBIAS COLDING: Right. So it's 1 over this here. But remember how r was defined. r is up there. Yeah. OK.

Now let me just say a word about why does the proof from last time, why does it really prove this more? I mean, the argument we gave last time, why does it give you this? I mean, it's much more satisfying, this version of the root test, because it applies to any series. Because otherwise, we talked about whether some ratio was convergent, but it may not converge. So this is much more satisfying.

So suppose, again, you have a series like this. And so now you're looking at limsup. So this is, again, this is the root test. Root test version. Let's call it version 2.0. And so you take a series serious like this, and it's looking at limsup, and then the n-th root of the absolute value of the n.

And again, it's saying that if this thing here is less than 1, it's convergent. If it's bigger than 1, it's divergent. And if it's equal to 1, it's inconclusive.

Now, let's try to see why this here is the case. So proof of root test version 2.0. Maybe I won't do all of it, but let's just look at. Because in a minute, you see that it's really kind of super similar to last time.

So suppose that this limsup of the n-th root here, of d-n, suppose that this thing here is equal to some d, which is strictly less than 1. OK? So this means that there exists-- so if you will, choose d bar with d less than d bar strictly less than 1. So I can choose it like this.

But now you see that if I choose it like this, then there exists-- so remember, let's just remind ourselves that-- how was the limsup defined? The limsup was defined to be that you take this new sequence--

So you're taking this sequence here. So you're taking these guys here. So you're looking at now these guys here, and you define a sequence that is like the sup, starting at the n-th guy, and like this here. So this here is n plus 1, et cetera. Right? This is the sup of the tail. And so that was sup on-- as you're going further out, you're taking sup over smaller sets. So this sequence going down.

So now if this limit here is sup d, which is strictly less than d bar, then it means that they exist. So there exists capital N. So that if little n is bigger than capital N, then all of these guys here is less than d bar, right? So if this here, then all of these guys here is less than or equal to d bar. This is if little n, if you're starting that far out, right?

So you have that when n is bigger than this, you have that this. So you have that for all these guys here, d-n, the n-th root is less than d bar-- less than or equal to d bar. But now you take the n-th power on both sides, so you have that the n absolute value is less than d bar, the n power.

This here is, d, this isn't even actually non-negative, so you don't even need this absolute value sign. So you're having this. But this is a number that is strictly less than 1, so this here is a geometric series. Geometric series where d bar is less than 1. So it is a convergent geometric series, right?

So this is, you see, that it's basically the same proof we did last time. But last time, we were assuming that some ratio was converging or we were assuming that this here converges to something, but you don't need it to converge. You just need that the limsup is less than 1.

And so likewise, if you examine what this case here say, then just like last time, it's going to say that if this here is bigger than 1, then the ordinary element doesn't even go to 0. So it fails the most basic test for being convergent And that was just like last time, also.

And again, the advantage-- I think you were the one that was asking yesterday. I mean, last time this was the ratio test and the root test, and like in the ratio test, it's something about that the next element is going down with some fraction. And you very much see that this here works for all series. And it's really just because you don't compare the element to the next, but you're comparing it to this fixed. series, which is a geometric series. So that's the real version, so to speak, of the root test.

So now let's talk a little bit more about power series. So like this one function that is particularly important, and I mean, all of these basic functions are, of course, important, but they are somewhat similar. And the one I want to talk about is the exponential function.

And there's various way that you could define, so what is the exponential function? There's various ways that you could define the exponential function. You can define it, like, two that comes to mind. But there's others, too. One is that you could define it to be a function that's satisfying a certain differential equation, and that would be a good way of defining it.

Another way of defining it, which is more relevant to what we're doing here, is defining it as a power series. So I'm looking at e to the x, and I'm looking at e to the x. So I define E, capital E of x, as this power series that we looked at before.

Now I want to prove that this here is really what you should think of. This here is the exponential function. But what does it mean that is is exponential? I want to show that this is the exponential function. So what do I mean by? I mean, of course I could define any function to be the exponential function, but there are certain properties of the exponential function that you would want.

Well, the first thing I observe is that E of 0 is equal to-- if x is 0, then all of these are 0, except when n is equal to 0. And then x to the 0, this here, is, by definition, one. And this is 0, factorial is by definition one. So this here is 1, by definition.

And we know already, by the way, that this here we have already seen that the radius of convergence for this power series, of convergence is equal to infinity. We have already seen that, so it's a well-defined function for all x.

So next time we will show that if you define it like that, that this function here, defined like that, has a property that x of E to x plus y is equal to this product.

Well, the first thing I can observe is-- I haven't really gotten to the property yet, but at the moment I'm just examining that particular function, some properties that it has. And so the first thing I will observe is that if x is positive or non-negative, right, then all of these numbers are non-negative. So you're summing non-negative numbers. So the sum is non-negative.

So if x is bigger or equal to 0, then it implies that E of x is bigger, is actually strictly positive. In fact, the smallest one is where x was equal to 0. But in that case, you got one, right?

Now if x is negative, then it's not super clear. But if x is negative-- some of these numbers that you'll be summing are positive and some of them are negative, so it's not at all clear what happened to the sum. But if you grant me this for the moment, this relationship here, then we would have that E to the x minus E-- minus, sorry, E in x minus x--

So that's the same as-- so E of 0, which I already know is one, this here can think about that as x minus x. Or, if you will, to think about it as x plus minus x. And now, if you grant me this thing here that we prove next time, then I can write this as E of x times E of minus x. Right?

And now if x here is negative, then of course minus x is positive. So now I know by this thing here that this here is positive. And so, since the product of these two things is 1, this here must be strictly positive as well, right? So this implies that E to the x is also strictly positive. So this is just a couple of observations. Now, of course, I'm using that relationship that I haven't established yet, but I will establish that next time.

The next thing is that I want to define the number E, so E, that is 2.7, whatever. I don't remember. So E we define-- so little e, so I define it as this function here evaluated in one. OK.

So I have my function, capital E, but I want to define a second function that is e. So I want to define a second function, and that is e to the x, like this, and that should have the kind of properties that you think it should have.

But first, before I do that, let me just think. If you take e to the power m here, let's say that m is an integer. This here should just be e times e m times. Note that this here, I can, of course, write as e. This is what e is, so I can write it as this here, m times. But now I can use this here that I will prove next time again.

So if I take E of 1 times E of 1, then that's E of 2. And if I take E three times, that is E of 3. So this thing here is actually the same as capital E of m, right? So I have that.

By definition, e to the 0, this here I'm setting equal to 1. But I already have that one was equal to this capital E in 0. Now what about if m here is a negative integer?

So suppose that m here is a negative integer. Suppose that m here is negative. What should e to the m here be? This here is just 1 over e to the minus m. OK, minus m is a positive integer, so this makes sense.

Now, if you're looking at e to the-- again, using this relationship here, e-- and in fact, the one we just did. So this is on integers, and this here is now equal to E of 0, which is equal to 1. And this is, by that relationship over there, it's E of m times E of minus m. And so you see that this implies that E of minus M is equal to 1 over E of m, by just dividing this on both sides.

So you have this, and this thing here, but this thing here-- sorry, this is not what I wanted. If I take m negative, I just use the other one. Sorry. E of M is equal to 1 over E of minus m, like that. M was supposed to be negative, so this means that minus M is a positive integer.

And this one here is then this one. I know that this here already agree with this little function of e. So you see that this thing here agrees with e to the power m. Because we define it like this, and now just checking that also on the negative integer, this function here and this function here agree. OK.

So I now have these two different functions, but one I haven't really defined yet. I've only defined it now on all integers. But so far, these two functions agree.

Now what I also should have is that I should have-- so suppose you're looking at a rational number. So suppose you take a rational number. Remember, we usually write a rational number. So suppose that q here is a rational number. We usually write a rational number as an element in set. So it could be negative. It could be 0. And n here is a positive integer, so a fraction like this.

Now what should e of q be? Well, e of q, this here should be-- how you would think about it is that it really should be something. This here should be a number. Should be a number, alpha, and this should have the property that is positive. And it should have the property, so let's just write it.

So e of q, this is m over n. It should have the property that if you raise it-- alpha here, suppose you were raising it to the power n, then you should get this number that we have already defined, right? So it should be positive n when you raise it like this, and then it should give you that. And just like with the square root of 2, there's just one element with this property. OK, So that's defined.

So you see this way we have now defined this function, e, on all rational numbers. But now we need to check. We need to check that-- this thing here-- we need to check that e of q is equal like that. So I need to check that.

So I want to say that e to the q is that alpha, right? So this means that I want to show. So I need to show two things, e to q is positive-- well, I've already checked that. We checked that E, capital E at anything is positive, so this is OK.

The second thing I need to check is that it has the property that if you raise it to the n power, then you're getting e to the m, right? Now e to the m, that's an integer. It could be negative. But e to any negative integer, whether it's positive or negative, we already checked that this is equal to capital e of m.

So now I need to show this here. So now what is this? So what do I need to check? So I need to check that e to the m over n, this thing here, raised to the power n is equal to this. But what is this here? This here is just e to the m over n, multiply itself n times. So this is now n times.

But now I can use, again, this thing that I proved, next time, that when I do it like here, I'm just doing it 2 times, and here I'm doing it for different x and y. But if I do it n times with the same x, then I'm just adding that x together n times.

And so this thing here is the same as e, by that rule over there, n times m over n. But this is just E of m, as it should be. So you see that these two functions here. So the conclusion now is that this function here is equal to this function here for all q for rational numbers.

Now, what other property should this exponential-- so this here, so far, this is kind of the property that you want of the exponential function. But what other properties should the exponential function have? You should have one other property, and that's enough to determine it, and that is it should be continuous.

So one more property is that this function e to the x, when x is defined like this, this function should be continuous, and we'll talk a little bit more about continuous function in just a second.

Now that I have a theorem that we'll prove next time, one of the next times. And that is that, suppose you take two functions, so we then can appeal to the following theorem. If f and g are two continuous functions on r that agree on all of q.

So if you have two continuous functions, and if you restrict it to the rational number, then the two functions are the same. If they are both continuous, then the function has to be the same. So this implies that f is equal to g everywhere, not just on the rational function. No, rational numbers. So we prove that either next time or the following time.

Let me just go back and remind you about what it means for a function to be continuous. So this is the following. So continuous. So suppose you take a function, and the function, it doesn't even need to be defined on an interval. But you may as well assume that it's an interval, but it doesn't need to be defined on the interval.

So a function here, f, is said to be continuous at x here in the domain where it's defined. If for all epsilon greater than 0, there exists delta greater than zero, such that if x minus x-0 is smaller than delta, then the images are epsilon close to each other.

And so again this delta typically depends not just on epsilon, but also on x-0. So it's continuous at a point x-0 if this is the case, and the function, f, here is said to be continuous if it is continuous at all points.

Now let's just look at two simple examples, and then once we have those, then together with some algebraic rules that are similar to the algebraic rules that we had for sequences, then we get a huge number of examples. We get a large collection of examples of continuous functions.

So the first example is the simplest of all examples, where f here is the constant function. So I'm thinking about f here is defined from R into R. And for each x in R, it's equal to the same constant. So that's f. I want to show that this f is a constant function, is continuous, so proof of continuity.

So given epsilon greater than 0, at this point you can pick any delta you want. I'm just picking. So let delta be equal to 1, but you could pick anything you want. It has to be positive, but you can pick anything you want. Then you have that if x minus x-0 is less than 1, well then of course, f of x minus f of x-0, both of these numbers is c. So the difference is c minus c, so this is 0. So this is surely less than epsilon.

So this shows that the function is continuous at all points. And you see that you could pick delta anything you want, and, of course, in particular, it doesn't even depend on x-0. It doesn't depend on epsilon, and it doesn't depend on x-0. So the constant functions are continuous. What about the next symbol?

The next simplest function would be the linear function, or just x function, x. So suppose you're looking at the function f on R to R, and where f of x is equal to x. So this is also continuous. Also continuous.

And the proof of that, given epsilon. set delta equal to epsilon. Then you see that if x minus x-0 is less than epsilon, then f of x minus f of x-0, but f of x is equal to x, and f of x-0 is equal to x zero. So this thing here is, of course, less than epsilon, right? So that's kind of trivial.

Now the next thing is that we have the algebraic property of continuous functions. So we have some algebraic property. Properties of continuous function. And I will prove a couple of these, and the proofs you'll see in just a second. And there's good reasons for that. They're very similar to the property we had for limits of sequences, and the proofs are very similar to those proofs.

So the first one is that if f and g are continuous, then the sum of these two functions is also continuous. The next one is that if f is continuous and c is a constant, then C times f is continuous. The third property is that, again, if f comma g are continuous, then the product f times g is continuous.

The fourth property is that if f is continuous, and f is never equal to 0, then one over f is continuous. And the fifth property, that's a little bit different. We didn't have that for sequences. The fifth property is that if you take a map, f of g, take a function, f, you think about it as a function from R to R, and then you have another function, g, from R to R.

And now you could look at where you check first. You could look at the composition. So you can look at the function h of x, which is where you first apply f to x, and then you apply g to it. So if both f, so if this here and the function g are continuous, then the composition here is also continuous.

When you have a composition of function, you always have to be careful that it makes sense. So to make sense means that the function, g, the one that-- this thing-- function that you apply, is that the image of-- when you use the first function, you have to make sure that what you get here is at some point where g is defined.

So just for simplicity, I just looked at maps, into functions into R where g also was defined on all of that. But of course, it doesn't have to be defined, but you just have to make sure that g is defined where f and x end up.

I'll talk a little bit about how to prove these, but it's very similar to what we did for sequences. Let me just apply it, so let me look at an example where I apply it. So suppose you take x squared plus 1. Suppose you take this function here. I'll take a look at two examples.

Suppose you take this function here. I claim that this here is continuous. I'm not going to go through the proof because we already-- actually, so this here proves at least some of these properties. But once you have that, then you have that g of x equal to x. This here is continuous. This is what we proved just a little while ago. And if you take the function that is constant, and in this case constant 1, we already proved that that's continuous.

Now this means that if you take a continuous function by whatever, rule three or something like that over there, and you take the function where it's multiplied by itself, this function here. Because g is continuous. This function here is also continuous. And then once you have that, you can add it, g squared plus h. You now have established that this is continuous. This is a constant function, so we already proved that this is continuous. So the sum, by rule one over there, this here is continuous.

So this show that this function here, this quadratic function, is continuous. But in fact, of course, this argument gives you that any polynomial, any polynomial is continuous. So this here proves that any polynomial-- So if you're using 1, 2, and 3 then it's just starting at the constant function. And the function x, you can get to any polynomial, and using these operations. And so this means that any polynomial is continuous, right?

So you have the following theorem, all polynomials. are continuous. And in fact, so this is just using 1, 2, and 3 over here. If you now also bring in four, then you have-- so here is proof. So it uses the two examples that we gave, and then in addition to that, it follows from the algebraic rules 1 through 3.

If you bring in also 4, 5 is something slightly different. But if you bring in 4 here, then you get what's called any rational function. All rational functions. are continuous. So what is a rational function? A rational function is just where you're taking a polynomial, and then you're dividing it by another polynomial. So we know that any polynomial is continuous.

If you take another polynomial-- and let's look at that other polynomial away from the roots, away from where it is 0, away from where it's 0. I mean, it's of course even continuous where it's 0, but you can't divide by it, right? So away from it, this function here is not 0 and is continuous. This here is continuous, rule four, so this is proof that the theorem above plus rule four.

AUDIENCE: Earlier you were showing that the function of x shows your delta as a function of epsilon.

TOBIAS COLDING: Exactly. In that case, I chose delta to be simply epsilon. Yeah. Yeah.

AUDIENCE: And you can do that?

TOBIAS COLDING: That's not a problem. If you give me an epsilon, then I have to pick a delta. And typically, the delta will depend on epsilon. Typically, it also depends on X. Let me give you an example of the last. I might not get to proving this, but that's fine. I mean, they're actually the proof. I'm not even sure I would go through it. Maybe if I have 10 minutes, I will talk about one of them.

So let me give you an example. So typically, it's highly unusual in the first example with a constant function that it doesn't really matter. You could pick delta anything. Now suppose you're looking at the function f of x equal to 1 over x. This is a rational function, because it's a polynomial. In that case, the polynomial in the denominator is the constant function, 1, and then the denominator is x.

Let's just look at this function where it's strictly positive. And then you see now the graph looks basically like this. So in particular, if you go close to 0-- it's not defined as 0. But if you go close to 0, then the image varies a lot.

So if you take some fixed thing here, suppose you want to prove that it's continuous at some x zero. So this means that you have to pick some delta. So given an epsilon, you have to pick a delta so that if you are in this little interval, then the image lies in that little interval. But you see that as x-0 becomes very close to 0, then in any small interval, the function varies a lot.

So this means that if you take some fixed epsilon, then you expect that if the x-0 was close to 0, then the delta had to be really, really, really small for this to work. So typically, delta depends not just on epsilon-- which, I mean, it's highly unusual it doesn't-- it also depends on x. And in fact, if it doesn't depend on x or x-0, then it has a particular name. We'll get to that later. It's called uniform continuity. Uniform continuity has some nice extra properties.

And you will see later on that there's a good reason why the domain here did not include 0. If somehow this would have been a continuous function defined on an interval that included the end point, then actually it would automatically be uniform continuous. So it was a good reason. This example here, it didn't include the point 0.

So I have three minutes left. Let me just talk about one particular of these. Like, the first one and the second one is super easy, and the third one is also easy. And it uses exactly the same trick that we looked at before, so let me just talk about that for a minute. Just because it's a little bit more interesting than 1 and 2, is barely interesting. So suppose I'm looking at the product of these two functions.

And so remember that you give me an epsilon, I have to find a delta. So I have to find a delta so that this difference here is smaller. I want to find the delta, so that if x minus x-0 is smaller, if it's smaller than delta, then this difference here is smaller than epsilon, right? That's what I'm supposed to do to just knowing that f and g are continuous.

And so just like when we had this for sequences, the similar thing for sequences, it's too complicated if you just look at it like this. So what you do is that you take f of x times g of x. You insert something, and it doesn't really matter which way you do it, but this is halfway between this and that where you take one point to be x-0, the others are x. Of course, you didn't have that, so you have to add it again.

But now the thing is that what you've done here is you have split the difference, if you will. And so now when you want to estimate this, you estimate this thing here first. So this is g of x times f of x minus f of x-0. And you take the absolute value of this, and then here you can factor out f of x-0 and g of x minus g of x here. Now, you can do like we did before. This here is now fixed, so you can also take the absolute value sign like that.

And now x-0 is fixed, so this is just some fixed number. So this means that this thing here has to be maybe even smaller, depending on what this constant is. But that's easy to achieve with some delta, right? Here, it's a little bit more complicated because this one here is not fixed. But you have to then say that once you are close to x-0, then this here is bound for this. You see, it's like what we did before. So we're out of time, so I'll just leave it there.

So next time we'll talk more about this with exponential function. Yeah?

AUDIENCE: So our rational function is said to be continuous or do we not call them continuous?

TOBIAS COLDING: No. Any rational function is continuous. It's defined at points where the polynomial in the denominator away from the roots of the polynomial in the denominator. It's continuous. What?

AUDIENCE: It's said to be non-continuous if the roots of the polynomial [INAUDIBLE]?

TOBIAS COLDING: What? Yeah, I mean, any rational function is continuous. It's just not defined at the roots. OK, now for some polynomial, some rational functions, if you were silly and you wrote this thing here, this is a polynomial. That's a polynomial. This one has a root at 0, but really, this function here, you would probably think about as being the constant function 1. But it's not a pure-- it's continuous away from the root. It doesn't say whether it's continuous at the root. It's just not defined at the root of the denominator. Yep.

AUDIENCE: [? One ?] point is not like that. You didn't say continuous or [INAUDIBLE].

TOBIAS COLDING: Yep. That's right. That's right. That's right. So if you have a polynomial in the denominator, there's finitely many points where you have roots. And at those points, it's absolutely not defined. You might be able to extend it in a continuous way to those points. But typically, it's not defined in those points.