[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: Just begin with-- so if we have a metric space here, then a cover of x is a collection of subsets u alpha so that the x is contained in the union of u alpha. So that's a cover of X and likewise, if a here is a subset of x, then u alpha is a cover of a if a here is in the union of the u alpha.

So that's what it means to cover the whole set or just a subset. Then there are some covers that are particularly important, and those are the open covers. So a open cover is a cover that consists of open subsets.

And then finally, we have that, so if again, x,d is a metric space, a here is a subset of x, then we say that a is compact if every open cover of a has a finite subcover, OK.

And one thing we proved last time, was that if you have-- a so we proved last time, that if you have a here is compact, then a is bounded and closed. Right, so it's a is a closed subset, meaning that the complement is open, and a itself is contained in some large ball centered at some point in x.

Now, the converse is not the case always. So In Rn the usual metric, Euclidean metric, the converse is the case but in general not-- but not in general.

And so we looked at one example on this last time. So we looked at the following example. But it was a rather trivial example that if you take x here-- if you take the space x to be, say, the interval from 0 to 1, and the metric here, this here is the usual metric, so the distance between two elements here is just the norm of the difference.

And then well, x is the whole set, so with this here, x is the whole set. So x here, if this here is the metric space, then x is, by definition, closed because the whole set is closed. It's not closed as a subset of r, but if the metric space just is x, then it is closed, and it's bounded-- and bounded.

In particular, this interval here is contained in the ball of radius one half around one half. But it's actually equal to that ball, if you will. So it's definitely bounded, but it's not compact, right? So it's not compact.

Because you can write x here as the union of 1 over n to 1. And now you take the union over all positive integers. But if you only take finitely many of them, then you don't get all the way down to zero, right? So it's not a union. It's union of all of these. Each of these r is open, But. There's no finite union of those sets that cover x.

A more interesting example is that-- suppose you take-- suppose you take-- so here's another example that we didn't get to last time, that if you take the continuous function on the unit interval, and so if you take-- and now we use this metric that we have looked at before, where the distance between two functions is the max over x in this interval here of the norm of f of x minus g of x.

We've already seen that this here is a nice metric. And if I now look at-- in this metric, I look at the unit ball, so I'm looking at the closed unit ball, which means that I'm looking at the set of functions here, so that the distance of f to the function that is constant equal to zero, this distance here, is less or equal to one.

So I'm looking here at the subset of the continuous function on this interval, so that the distance to the zero function is at most one. So let me just write this out. So this means that this is the function f in this continuous function here, so that the max of f of x minus 0, so I don't really have to write 0, is less than or equal to 1.

Now, this set again, this set here, is what we denote by the-- this is the closed unit ball because you have less than or equal to here. And the closed unit ball, the open unit ball we have written, this is now the function that is identically equal to 0. The open unit ball, we wrote like this. This is if you have strict inequality. And to denote that it's less or equal to, I'm just putting a bar here.

And so we already know that this thing here, if you take a metric space, and you're looking at a closed unit ball like this, we already proved that this here is closed. Because we proved directly, using the triangle inequality, that the complement is open.

OK, so this is a closed subset. It's obviously bounded because it's actually even just a ball, right? But I claim that-- so it's closed. It's bounded, but it's not compact. And here's a interesting example of a sequence of elements in this ball.

So I will now construct for each n, which is a natural number, we construct a function fn in this that is a continuous function on this unit interval. And the function f will be given as follows. It will be one from when-- let's see. Let me make it 0/ I guess it doesn't matter. I'll make it one when x here is between 1 over n plus 1. No, no, I would make it equal to 1 when x here is bigger or equal to 0 and less than or equal to 1 over n plus 1. Then I'll make it when x here is bigger than or equal to 1 over n but less or equal to 1, then I'll set it equal to 0.

And then you see now, I wanted to do so-- so let's look at it. So this function here is if you draw this function-- so I have here 1. Here I have 0. And this function is 1 up to 1 over n plus 1.

Maybe I'll draw it a little bit further out. Let me actually make the picture bigger, so like that. Here I have 1. Here I have 1 over n plus 1. And here I have 1 over n. And the function is supposed to be 1 here. And then it's supposed to die off between here and here. And I choose it to die off linearly. And then it's 0 here.

So if it's going down linearly, this means that I have my-- what is the difference between these two? It's 1 over n minus 1 over n plus 1. This is this distance here. And so this here, putting it on a common denominator n plus 1 over n times n plus 1. This is what you get from this. And if you put it on the common denominator for the second one, this is what you get here. And so this is just 1 over n like this.

And so this is this distance here. And so if I define the function to be-- so I'm setting it equal to this. And now in between when x here is bigger or equal to 1 over n plus 1 and less or equal to 1 over n, I will set it equal to 1 minus n times n plus 1 times-- sorry, let me make more space. Yeah, let me make more space.

So here, this here was when 0 is less or equal to x, less or equal to 1 over n plus 1. This is when x here is between these two numbers. And this here is when x is larger than 1 over n And it's x is always in the interval between 0 and 1. And here, I'm choosing it's x minus and then 1 over n plus 1.

So you see that if you take x to be 1 over n plus 1, then here contributes 0. So at this left point, this here is equal to-- it's the function is equal to 1, as it should be. And when x here is at the other endpoint, then this here becomes exactly 1 over this, So this whole thing becomes 0. So this function here I drew, is if you write it down precisely, this is what the function is.

And now if I'm looking at the difference between two such functions, so for the next one-- you see the next one here? The next one here is-- so this here is fn. This is how fn looks like.

If I now look at fn plus 1, this start off being it's 0 to the right of this thing here. And then it's 1 once you get to n plus 2. So the next function looks like this. I don't know if there's any color chalk, but this thing here, that's the next function. So the first function is this function here. That's a graph of the first function. The next one is this one here.

And so if you're now looking at the distance between fn and fn plus 1, the distance here was the sup of the difference of the two. This here was just a max of f of-- like this. It was just a max of this, right? And now I know that these two numbers is always between 0 and 1. So this max here is at most 1.

But if I now evaluate it, if I'm now looking at-- if I now just evaluate it, if I'm looking at fn, and now I just evaluate it in this point here, in this point 1 over n plus 1, so if I'm looking at this thing here fn plus 1, 1 over n plus 1-- you see this value here, this first function here, and at this point, it's 1. This here is 1. But for the next one, you see at that point it's 0. So this means that this max here, is for all n and n plus 1, is actually equal to 1.

And so now if you look at the balls-- so now, in particular, if I'm looking at the balls of radius one half, suppose I'm looking at the ball of radius one half around-- so this is an open ball, so around fn. Because this here is an open ball, so it means that's strictly less than one half, so this means that fn plus 1 is not in that ball. So it's not in. So it's not in that ball.

And so now it's easy to see that all of these balls here are disjoint. So all of these balls here will be disjoint. They're all disjoint. And so if you're looking at if you can write this initial ball, this ball here, this was this close ball that we started with in the beginning, you can write it as a union. You can write it, of course, as a union of half balls where you take the union over all centers, all possible centers like this.

Because it's clearly cover this set because it contains all the centers. But that's already the whole set. But using this here, you can see that there is not-- that you can't have finitely many of these half balls that cover the whole set.

So using the functions fn, it follows easily that finitely many of these balls does not cover. And let me just cover this here.

Why is this the case? Actually, sometimes you put the bar all the way across. But I haven't done that. OK, suppose for a second, that you have-- suppose that you have a ball here of radius one half around some point. And suppose that you have a function, g-- two functions g and h that are in that ball, just using that it's a metric space.

So now we have that this thing here, g, is strictly less than one half, and the distance from f to h is strictly less than one half. So now using the triangle inequality, we conclude that the distance between this pair here, g and h, is less than or equal to the distance where you go over f. And so this has strictly less than one half plus one half, so this is strictly less than 1.

So you see from this, in particular, that if you take such a ball here, then you can't-- so it's impossible to have that fn, n fn plus 1. It's impossible to have that. They both are in that ball. Because if they were both in this ball, then the distance would be strictly less than 1. And we've already seen that distance is 1.

And this holds, not just for fn and the next one, but it holds for fn-- the same argument gives that from fn and any n larger than n, this here would be impossible. And so from this, it follows that you can't have finitely many that cover the ball, Yeah?

AUDIENCE: How do we know that the union of all of those functions and the one?

TOBIAS COLDING: Yeah, so this is just-- and this is important because we're going to use this in a minute anyway. So again, if you take a set here-- so if you take a metric space, and suppose you take a set, a subset of x, then you can always find a cover, even an open cover.

And in particular, in a minute, we will look at-- so if you take a here, so one example of an open cover of a, is that you're looking at the family of balls. You just pick some radius r as positive, And then you pick centers. But the centers-- sorry. You take the centers, x here is now anything in a.

R is fixed, It's positive. X is anything in a. Well, the union of these guys, the union certainly contains the centers. So this here already cover a. And we will use this. This is like a standard way to get a cover. And then if it's compact, then finally, many of those should cover.

So in fact, let's get to that. So the next thing I want to prove, is this Bolzano-Weierstrass version for a metric space. So you prove a Bolzano-Weierstrass theorem for metric spaces. Remember that the Bolzano-Weierstrass does not hold for every metric space-- for a metric space. So now, the Bolzano-Weierstrass does not hold for every metric space, but it holds-- so when this space here is compact.

So that's what I want to prove. And that's one of the motivations here for the definition of being compact. And so let me just remind you. So on Rn, on r, say, what is the [? Balzani? ?] So on r the Bolzano-Weierstrass theorem, what did that say?

So this is that any bounded sequence has a convergent subsequence. That was the Bolzano-Weierstrass theorem. And so what we want to show here, is the following theorem for a general metric space.

So we want to prove the following theorem that if x,d is a metric space, a here is-- subset is compact. Then any sequence in a has a convergent subsequence.

Now, let me try to divide it into two-- let me try to divide it into some subpieces. So I begin with a lemma. So I want to begin with a lemma that is the following. So I have a my metric space. I have a here is the subset that is compact. And then I have as a family of nested subsets, so let me just talk about nested subsets for a second.

So I'll do this on this board. So what do I mean by nested subset? So again, I have some set, and then I have some subsets, c. It's a family, but it's now a family that is indexed by the integers. So Ci here, is subset. So this is i equal to 1, 2, and so on.

In this case here, I am thinking about that the next subset is smaller than the previous, smaller or at most equal to the previous. So I have this. I could also have looked at where they were kind of increasing the subsets, but this here is what I have in mind. So the sets as I become larger, the set becomes smaller.

So now back to the lemma. I have this metric space. I have a compact subset. And then I have a sequence here of Ci's is a sequence-- is a nested sequence of subsets. So again, this means that the next element is contained in the previous one.

And this nested sequence of subsets, it's actually a nested sequence of closed subsets, closed subsets of a. So they're closed in x, and they're contained in a.

And then I am assuming that-- and assume also that the Ci's-- that none of the Ci's are the empty sets. So I don't want any of them to be trivial. Then the claim is-- so the claim is that if that's the case, then actually, if you take the intersection of all of these, this here is non-empty. So there is something that is in all of the Ci's. That's what I want to prove.

And in fact, this is quite straightforward. And because since the-- so the proof here, since Ci is closed, this means that the complement here is open.

And now for the complements here-- so the complement, let's call it-- the compliments, let's call it Bi. So Bi is open again. And Bi plus 1, they actually have this property that they are because the previous ones was nested where they became smaller. Then when you're looking at the complements, then they become larger.

So since this is closed, then we have this. If the intersection of these Ci, if there was nothing in the intersection of all of them, if this here was the empty set, but then it means that the complement of these guys would then cover a.

It would actually bind the cover because there's nothing-- the union of the complement. So the union of the complement here. They would actually cover x, but it would in particular cover a. So this here called bi. So you have that. So this mean-- and again, this here, this is the same as bi.

So now since a is compact-- since a is compact, this means that a here is contained in finitely many of these. Let's call them-- so you have finitely many like this of the B's. And I'm not sure whether it's the first, but there are finitely many of these that cover it.

And so in particular by one of these is like by one of the indices here is the largest one of those. And so this means that it's actually contained in the largest of these. Let's say that the largest of these is Bn. So you have this for some n.

But this here, remember that this here was the complement of Cn like this. So this means that a have nothing in Cn. But a was a-- Cn intersected with a. It's empty. But Cn was contained in a. So this means that since Cn was assumed to be contained in a, these were subsets of a, then this means that Cn-- so this means that this here then implies that this thing here, is this Cn. So you have that one of them was empty, but that we assume that none of the sets of Cn was empty. Was that clear? Sorry, Yeah?

AUDIENCE: Could it be like a sequence of Cis?

TOBIAS COLDING: That's right. So I'm trying to prove this theorem here. And it's not super complicated, but it's a little bit involved. And so I'm trying to-- of course, this is always a great idea, and it's kind of like an important thing in this class, and also if you take other more advanced math classes, is that when you're trying to prove something, there are things that are really easy to prove, but then there are things that are much more complicated.

So one thing is that you try to prove it. Another thing is that you try to explain to other the proof. And so in either case, if you can divide it into substeps-- and so this here is a substep in the proof.

And so in a minute, we will see why this here is relevant. And so this answers your question. I mean, how does this come up? And in a minute, you'll see how it comes up. So let me get back to that question in just a minute.

AUDIENCE: Are you able to explain the last remark again?

TOBIAS COLDING: Yeah.

AUDIENCE: [INAUDIBLE]

TOBIAS COLDING: So again, the Bi's are open and using compactness because all of the Bi's cover a then finally minimize coverage. But the Bi's are nested in this way that they are increasing as subsets.

So if I'm thinking about maybe this here is the one where the index is large, this then contain all the other. So it's actually equal to in that case. It would be equal. This really is equal to this.

So this means that a is contained in the complement of Cn. So in other words, there is no element in a that are in Cn. That's what it means. So there is no element in a that is in Cn This thing here, is empty.

But the Cn's, remember that the Cn's was assumed to be subsets of An. So this means that there is nothing in Cn that is not in a. So this is really just equal to Cn.

AUDIENCE: So is that essentially just creating a contradiction?

TOBIAS COLDING: That's right. And that's a contradiction because now let's show that this one here, this Cn is empty.

AUDIENCE: [INAUDIBLE]

TOBIAS COLDING: But they were all assumed to be non-empty, yeah, OK. So this is like-- yeah, sorry,

AUDIENCE: I have a question about-- when you think it's a contradiction, you said If it is the case that the intersection of the Ci's is, in fact, empty, then is contained in them. Why?

TOBIAS COLDING: OK, so if the intersection of the Ci-- so here, sorry. So I want to prove this. So I'm assuming that this is the case. And I define the Bi's to be the complement of the Ci's. So if there is nothing in the intersection of all of the Ci's, then it means that the union of the complements must cover the whole set.

So if you take anything-- even in x, if you take anything in x, it's not in all of the Ci's. So this means that there is some i so that it's in the complement of that Ci. But that's exactly this.

AUDIENCE: OK.

TOBIAS COLDING: So now you're asking why is this relevant? And that's a good question. And I will answer it in a minute. But I will kind of elaborate. I will give you an example where I have it, and then we will eventually see how to get to that example.

So the next thing is that suppose I take a-- suppose I have the same setup. But now my sets are the closed sets. So I have this setup again. So this is like a corner Of the lemma that we just proved.

So we have x, d. This is metric space. And we have a here is a subset of x. Ut is compact. And now my sets Ci's is going to be-- so Ci's here, these are going to be balls of radius Ri and center Xi. And there will be balls like this.

And I'm assuming that they are nested. So in the same way, I'm assuming that this thing here, the centers change. The radii and the center change is depending on i. They are all these closed balls, where it's less or equal to the i. So I'm assuming that the next one is contained in the previous one. And then I'm making one more assumption is that the Ri goes to 0.

So I have that. So just let's observe that these sets here, they are nested by assumption. They are closed. Because it's this closed ball. So they're closed subsets. Also observe that each of these balls contain the center. So this means that each of these balls is not empty. So it's automatic that they're non-empty.

So now I claim that the intersection of all of these balls is just one point. So now the claim here that the intersection of these balls here, these close balls here, is a single point. Yeah?

AUDIENCE: How do we know that each of the balls contain the center?

TOBIAS COLDING: Because if you take a ball-- so if you take a ball here, in any metric space, this is the y in this space, in this set so that the distance from x to-- now it's take the open one but similarly with a closed, this thing here is less than-- sorry, this here is 1.

AUDIENCE: It contains its own center but not one single point across all the balls.

TOBIAS COLDING: Yeah, so that's right. So I'm just saying that-- so if you plug-in x for y, you get 0. So this here x definite is in here. And I'm just saying that each of these guys is non-empty.

And then I want to claim that actually there is something in the intersection of all of these balls. And I want to claim that there's only one such thing. There's a unique thing that is in intersection of all these balls.

So now first, I want to prove that this thing here is non-empty. And this should have been the closed balls like this. But you see this just follows from the lemma. Because each B-- because each Bri xi is closed and non-empty. Each is closed non-empty.

And the family or the sequence of balls Is nested by assumption. By assumption we assume that this thing here is our i plus 1, xi plus 1. That this thing here is contained in the previous.

So by assumption-- so again, we use the lemma, and this is the Ci's. And the Ci's are closed, and they are indeed nested as they should be. And they are all non-empty just because they each contain their own centers. Yeah?

AUDIENCE: So by making that assumption that they're nested--

TOBIAS COLDING: Yeah.

AUDIENCE: --are we implicitly making the assumption about the structure like xi?

TOBIAS COLDING: That's right. We're making some assumption about-- right, we're making some assumption about that-- when you make this assumption about nested, there's some assumption about the centers, the distance between centers, and there's also something about the radii. How large is one radii relative to the other?

Because if you imagine that this here is Bri like this, and the picture would kind of contain the boundary, and then the next one-- but the next one may be out here on the side. But it's supposed to be contained in it.

So you see that the center may have moved. This here was the xi and the next center may have moved. It may have shifted quite a bit. But of course, this here is the Ri radius, and this here is the Ri plus 1 radius. So you see that there's some of relation.

So we have that it just follows from the lemma that the intersection of all these closed balls is nonempty. Yeah?

AUDIENCE: Doesn't it just follow from the assumption we made and that Ri goes to 0?

TOBIAS COLDING: Yeah, that's right. So it follows now from that i. That's why we're using that i equal to 0. Because suppose that you have two elements. Suppose that you have x here and y. Suppose that they are in the intersection of these guys here.

Now look at the distance between x and y. Well, they are intersections. So this means that for each i, for each i-- so for each i, we have that x here, and y is in this guy here.

So if it's in this ball here for an i, then you can use the triangle inequality. And this thing here is bounded by-- this here should have been Ri. This here is bounded by-- lesser or equal to because I have this bar --Ri plus Ri, so 2 Ri,

But if this here holds for all i, this goes to 0. So this means that this implies-- as you say, this implies that the distance had to be 0. But this is part of what it means to be a metric space that if the distance is 0, then the two points are the same. So the whole [INAUDIBLE] was really quite easy from the lemma.

And so now the question is-- so this is your question is, why is this relevant in this Bolzano-Weierstrass? I mean, they may be not so interesting, I mean, the lemma and the [INAUDIBLE] on its own, but there are tools to prove this Bolzano-Weierstrass theorem.

So let me try to first explain the idea. And then we go to the proof. So we have this metric space. And we have a subset. And the subset here is compact. A subset is assumed to be compact.

Then we have a sequence Xn. And the sequence here-- so this is a sequence contained in A. And we want to prove that there exists-- so I want to show-- want to find a convergent subsequence.

Remember that here-- I mean, just that there's no assumption-- I'm not talking about some Cauchy complete set space or something like that. So in this case, I really have to construct the-- I have to construct the convergence sequence. But I also have to construct the limits. And so this is where in particular, like the limit, this is why we want this thing here to have a unique intersection.

So that's what I want to do. So you have this space. You have the subset that's compact. And you are given a sequence. So and before I actually get to the proof, let me just try to explain the idea.

So again, this is before the proof. So I want to make an observation. So if I am in this iteration, and suppose now that I take-- so I have A here, and I take some r, sigma of r is being really small, and then do what we just did a little while ago. We cover. We're looking at A here is covered. You can write as A as a union of these balls here, where the center is anything in A.

So again, that cover A-- and so this means that finitely many-- that's finitely many cover A since A is compact. So I can write A here as contained in and then r-- now, I have to be careful. Because this is not the x. Probably, I should use a y here, sorry, just to not confuse it with the sequence that I have, OK.

And so it's contained in the union of balls like this, Ym. Y is contained in the union of finitely many balls like this. OK, so far we haven't used our sequence. But now, of course, I can say that we were given a sequence. We now have finitely many balls.

And there are infinitely many elements in the sequence. So it means that there is one of these balls, finitely many balls, so for one of them-- for one, maybe more than 1, maybe all of them. But for at least for one of these balls, there are infinitely many elements in the sequence.

Let me be very precise what I mean by that. Of course, the sequence could be like a constant sequence. What I mean that for infinitely many n's Xn is in one of these balls. So let me just write that out, Just. To be totally clear. So meaning for one of these balls, there are infinitely many n's such that. Xn is in that ball. And let's say that this is y3, whatever. So there's infinitely many n's.

So now, I can just think about that one ball. And I can just think about the sequence. Instead of the original sequence, I think about a subsequence that is contained in it, So this means that there is a-- therefore, there is a subsequence of the given sequence That is contained in this form.

OK, now, I want to somehow repeat this process. I want to repeat this process. And again, I haven't quite started the proof here, but I'm just trying to explain the idea. So this is like a step in the proof that you then for the right radii has to use over and over again.

And so I want to repeat this process. And for that, I need to remind you that if you take a metric space, if you have a metric space, so recall that if X here is a metric space, A here is a subset-- is a compact subset. A is a compact subset, and B here is a closed subset of X.

Well then A intersected with B. If it's compact, it's also closed. We proved that. And so this is the intersection of two closed sets. So this is closed. This is contained in A.

And now we proved last time that if you take-- so this here is a closed subset. This is closed. And it's contained in A. Since A is compact, if you take a closed subset of A, then this is also compact, so this is also compact.

So now let's continue that string of thoughts from over there. So now we repeat this. So I have a set. Maybe I should draw it even bigger. I have here this is my space X.

Then I have some subset. This here is A. A is compact. I covered it with balls here of radius-- and I now have that finitely many. These are balls of radius r around some y's. There's finitely many of these balls, and they cover A Then these sets here are, of course, not closed. And so I'm looking at this ball here, these finitely many. And I just take the closed balls.

Now I want to cover-- and I found that for one of them, there was a subsequence that was contained, in fact, in the open ball, so therefore, also in the closed ball. There was a subsequence of the original sequence. That's a ball I'm focusing on.

Now, this ball, I can then cover by balls of radius-- so I'm covering this-- so cover this by balls of radius. r over 4. So I'm going to write this thing here in the same way. I'm going to write this-- this here, I now know-- I'm only interested in things that are contained in A. So I know that if I intersect it with A, then this here is compact. I know that this is compact. That was because this here was closed. And A was compact. So this here is compact.

So this means that if I can write that set here-- I can write that set as-- so I can write this set here. So I start with A that is compact. Then I find a ball so that a subsequence is contained in that ball. This is with that ball.

Then I'm looking at-- so this is like the process I start with here. This is step one. Step two, I'm looking at this set. Three, I'm looking at now the closure of this ball intersected with A. This here is-- yeah, is compact. This here contains a subsequence of Xn.

This thing here is compact. This means that I can cover it by balls of radius-- so cover it by balls of radius r over 4. I want smaller radius, as I'm going along. So I'm covering it with-- and in fact, I can cover it by finitely many balls.

Why finitely many? This is because this is compact. I'm writing this here as the union of balls of this radius, where anything in this set is a center. And so I have the finitely many of those covered by compactness.

And so now what I observe, is that if I take-- and now I just had to be a little bit careful. So now I'm observing that if I take the ball of radius-- the double ball, suppose I'm taking this ball here, so I get finitely-- sorry, no, no, before that. So I have that it's now finitely many balls cover this. So this means that a subsequence must lie in one of these balls.

So this is the fifth step. A subsequence of the subsequence lies in one of these finitely many balls of radius r over 4.

And now I take-- So the next thing is I do-- I want to apply the collar from before. So I started with my space X. I got this ball here of radius r, and then there was some center. Then the next thing is that I get a ball of radius r. And now this is with one of these other finitely many balls, where a subsequence is called it's like this, maybe like one or something like that.

So the balls of radius r over 4, they cover this. So a subsequence action would have to lie here. But now I want to use the corner. If I need to use the corner, I need to have nested and I need to have closed sets.

And so in order to make them nested, then I double-- I'm going to double this ball. Why do I double this ball? Well, I double this ball here because these guys here cover this. But it means, of course, that there could be something outside. There could be.

So this is like a ball of radius r. And so then you have this ball of radius a quarter r. But the quarter balls, they may have some part outside and some part inside. So it's not this ball here. It's not contained in that ball.

But if you went to the double ball, if you went to this ball of radius 2r, you see this point here, the center here, it would lie-- there would be some points in this ball here that would be within. It may not be the center, but there's some point here that's within r of the center. And then by the triangle inequality, any other points is actually within r 2, r over 2 of that point.

So everything is comfortably-- so this thing here and even the half ball-- so I can conclude that even the ball of radius r over 2, even this one here. Is contained in the double ball like that. And in fact, if you make things less or equal to on both things, then you have that.

Now what I have is-- at least here you have something that is nested. And you see that the radius went down. So these are nested, and they are closed. And so now what we do is-- so now we want to do this for a sequence. And then of course, we want to-- so now we want to construct a sequence of closed balls like this.

And now when you pick your subsequence, you can't wait with picking the first element in subsequence. Remember, if you keep waiting, then you will end up getting nothing. So you pick-- at each stage, you pick one element in the subsequence. And then when you have the next ball, you pick the next element in the subsequent and so on.

And so we will do this with-- so we're going to do this procedure here. So now we're getting back to the proof of the theorem. And again, remember the theorem was this-- maybe I erased it now by mistake. But the theorem was this Bolzano-Weierstrass theorem for a metric space.

If you have a metric space, you have a compact subset, and you have a sequence in this compact subset. And you want to construct a convergent subsequence. So what you do is-- that's a proof of the Bolzano-Weierstrass for metric spaces.

So what you do is-- so you have this. Again, A here is compact, and you have a sequence Xn. This is a sequence that is contained in A. What you do is that first you can just let r be equal to 1. And then you do this. You cover it by-- you cover it. You have here A. And you cover it by finitely many balls of radius 1.

And for one of these balls, you have that there's infinitely many element in the sequence. So you're going to focus on that ball from here on and out. And you just pick-- in your subsequence, you now pick your first element in the subsequence to be some element in that ball.

So now the picture is that you just look at that ball. Forget about the other ball and everything else. You now cover it by balls of radius a quarter. And you have that-- again, that's finitely many cover-- this ball intersected with A, finitely many cover it. And so one of them has infinitely many elements in this subsequence you consider. And so you just pick one element in that. That's the next one.

And now you repeat this process. And so you see that for these balls-- but remember that-- so you're starting with the radius 1. But you have to double things. So the sequence of closed sets, the initial closed sets, will have radius 2. And the next one won't have radius a quarter, but it will have radius 2 times a quarter, so one half.

And the next one will then be a ball. It will have a different center, and it will have radius one half. And then the next one you see, at each stage when I do it like this, then it goes down with a factor of 4 at each stage. So this here would be 1 over 8 and so on.

And so you see that this sequence, these closed balls, they are nested as they should be. And the radii goes to zero. So this means that there is an element that lies in all of these, and that is now the limit because of the subsequence.

From here on and out, it would be in that ball and from it later on and out, it would be in the next ball. So that intersection of all of these balls will be your limit. Yeah?

AUDIENCE: [INAUDIBLE]

TOBIAS COLDING: So really what you're looking at that this thing here, it's intersected with A. So it's just that these balls here, so the intersection here, that these are the closed sets, and these are the compacts. They are each of them are non-empty. And they are the closed subsets of A. So you're really arguing on these guys here. You see what I mean?

I know I stated it for balls in the column. But if you just think about it, the balls play no particular role, except for saying that the intersection of all of them contain just one point. And that was because the radii went to zero. The important points were that the sets were closed subsets.

Because then there was the lemma was just all about closed subset. It had nothing to do with-- And so you see that these guys here are closed subsets of A. And so-- yeah?

AUDIENCE: I don't quite understand where the [INAUDIBLE]. Is it, you're looking at the ball of radius 1 and trying to see how to contain the ball of radius 1.

TOBIAS COLDING: You mean why I went to double the ball, no?

AUDIENCE: Yeah.

TOBIAS COLDING: OK, so the reason I go to the double ball, is that I want to make sure that things are nested. And if I just did this procedure-- suppose so again, I take this ball here. This is the first step. I have these balls of radius 1. And I find one of them so that there's infinite many in the sequence.

Then I want to have something finer. And so I go to-- I chose because I think-- you didn't quite have to do a quarter. You could actually have gotten away with a half, but I thought it wasn't as clear if I did it. So I did it with a quarter because then I thought it looked more obvious.

So if you now take this ball here, and as you say, you always have to intersect with A, just to make things compact. So I cover it with balls of radius a quarter. But some of these balls, part of it may lie outside this ball here.

Because I couldn't cover-- you see what I mean, if I take a ball here. I can't cover it by finitely many balls. Just if it was like Euclidean space, if I insist that the ball had to be-- this smaller ball had to be inside this ball, then there wouldn't be finitely many. So I have to look at balls where some of these balls is kind of outside the surface.

But now, of course, so that wouldn't really work because it's not like this ball here, let alone the double of this ball. It's not contained in this. So that's why I doubled that ball. Then everything is comfortably contained [INAUDIBLE]. Any other questions? Yeah?

AUDIENCE: Since they're all closed ball-- or wait, no. These are open balls?

TOBIAS COLDING: When I cover things, they are open because I have to have-- when I have a compact set, I have to cover it by open subset. But then once I'm looking at the nested things, I have to go to the closure, because otherwise the lemma and the corner would not apply.

AUDIENCE: OK.

TOBIAS COLDING: OK.

AUDIENCE: And [INAUDIBLE], if they were closed though, you would be able to cover like E1 with a--

TOBIAS COLDING: And I'm just saying that if the open balls cover it, then of course, their closure also cover it because the closure is a larger set.

AUDIENCE: Yeah.

TOBIAS COLDING: But it wouldn't be a priori clear that-- it's not like they're closed set. So that it's not working, the definition. Yeah?

AUDIENCE: I'm not sure if you have proof that up there we have that if you have a compact subset, and you take the intersection with a closed subset. Yeah, we proved that it's closed. Can we prove it is compact?

TOBIAS COLDING: We prove that it's compact. And I can prove it again because it's very easy. So if you have a metric space, this here is compact. And now you take a subset here of this here. And this here is closed. It's a closed subset of X.

Then you look at X take away B. You take any cover of this. Together with this, they cover A. But then finitely many-- but this here have nothing in B. So actually finitely many of the other one cover B here. Any other questions? OK, great, thanks.