[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: Last time we talked about the Picard-Lindelöf theorem. And so this theorem is about solutions to ODEs. And so, specifically, it's asking that-- so you have a function f and a function g. And f here is a differentiable. And the derivative of f, as well as g, are continuous.

And then the problem is that you're looking for solutions. So you want a solution. So looking for a solution y to the ODE.

And the ODE is at the derivative should be equal to f of y. So at some point x, the derivative of y should be equal to f of y of x plus g of x. And you are looking for that this function, that y of 0, say, is equal to a. So this is what's called the initial condition. And this is so-- this is equal to that.

And so in order to-- so we want existence-- so want existence and uniqueness. So existence just means that there is a solution for a given a. There is a solution. Uniqueness means that if you have two solutions to this ODE, then they will have to coincide.

And so the way we were setting it up was that we were using-- so using the contraction mapping theorem. So the contraction mapping theorem is the following, that if you take a metric space x, and this metric space is x, d is a metric space, is a Cauchy complete metric space then a contracting map-- so a contracting map is a map g from this metric space into itself so that-- and a constant c strictly less than 1, so that if you measure the distance between the images, the images are squeezed together.

So this means that the distance between the images is this constant that is strictly less than 1 times the distance between x and y. So this is a contracting map. And the contracting mapping theorem say-- so the contraction mapping theorem is the following.

There exists if you have a Cauchy complete metric space and you have a contracting map like this, then T has a unique fixed point. And remember that a fixed point-- so a fixed point. This is x in this space so that T leaves it invariant. And so we proved this. We proved the contraction mapping theorem last time.

Now, how were we going to use for this ODE? So we defined-- so if I here is an containing a 0, then we defined-- and u is a function. And u-- I'm sorry, y is a continuous function, then we defined g of y. So this is a continuous function on the interval on I.

Then we defined another continuous function on the same interval as follows, that's the T of y. So this is going to be a continuous function. So in x, this thing here is-- this is a. Because we want it to have value at 0 a, and then we're integrating from 0 to x of f of y of s plus g of s ds.

And so this-- this here gives you a map up from-- gives you a map from the continuous function on this interval into the continuous function on the interval. And it also has the property that if you take a function and you evaluate it at 0, then you get a. And now, we observe that, in fact, the image here-- the image is-- if you define-- when you define it like this, by the fundamental theorem of calculus, this function here is differentiable.

So this function is-- so it actually takes-- so T actually takes the space of continuous functions into the space of functions that are not just continuous, but actually differentiable, and even the derivative is continuous. And the ones that are-- so this here is the space of functions. So this is functions that are where the functions y so that y prime exists and is continuous.

And so now, what we saw last time was that by the fundamental theorem of calculus-- of calculus, the derivative-- so by the fundamental theorem of calculus, the derivative of this function, if you take y and you take T of it, this is now a function, and it's differentiable, and the derivative is f of y. So at x, it's f of y of x plus g of x.

And so this means that if-- so if y is a fixed point for the map g-- so if it's a fixed point, this just means that g of y is equal to y. Well then this thing here, this is the derivative of y. So then it means that y here solves the ODE.

We have already checked that the initial value-- we did that somewhere here. We already checked that the initial value is what it should be. And if it's a fixed point, then the derivative is also what it should be.

So we want to find a fixed point. And so what we did last time was that we saw that if we restrict the interval-- so last time, what we proved is this, that if we restrict the interval-- so last time we showed-- so we set. M here equal to a, absolute value of a plus 2, and then we saw that there exists a delta 0 at delta 0 greater than 0, such that if you restrict to the interval, if you're looking at the interval of continuous function, the interval of continuous function on this little interval-- so this is an interval around 0 and you're looking at this map, then this map here-- well, first of all, we already know that it takes this the space of continuous function into the space of continuous function.

But we have something more. We have that if you're looking at the subspace of this, which is the space of functions y that are here, and you also require that the norm here is less than or equal to M, then T takes this space here into itself. This is this lemma that we proved last time.

So this space here, you can think about this space here, as we did last time, as-- so then we defined-- we defined-- we defined capital R equal to-- no, actually. Sorry. We defined-- we looked at the balls just on this.

So we looked at the ball of radius capital N in this around this 0 function. So this thing here is the closed ball in the space of continuous function with the metric that we've been using. And this set here we can think about, if we want, as this set.

So now what we have is-- so we have-- so the setup now is that we want to find a fixed point for this map T. We have a space that we want to look at, and the space is, again, now we're going to think about-- so just to-- so this is the space we're looking at. But this is what we before have denoted. Like, the ball in this metric space, it's centered at the function that is constant 0 and this is the radius.

And so I'm now looking at this, this map. So we have now shown that it takes this metric ball into itself. And I want to find a fixed point. I want to find a fixed point. Fixed point of T.

So I need to-- I need to show-- so we want to show-- want to show that T is a contracting map.

Why? Because if it is a contracting map, then we already know that if you take a closed interval like this, and you equip it with the usual distance that we have, this here is Cauchy complete. But now if you take a Cauchy complete space, and you're taking a closed subset-- and this here is a closed subset-- then this here is also Cauchy complete. This here is also. So if we could prove that T from this space into itself is a contracting map, so it means that there exists a C strictly less than 1. Then we would have a fixed point, and we would have a solution to the array.

The thing is that this is not quite going to do it. It's not quite a contracting map. But so we will need-- so for it to be a contracting map, we need to restrict the integral further.

So by that, I mean that we need to find-- we need delta to be-- we need to show that there's some delta. It will actually be less than. So we need to show that there is some delta that is smaller than this delta zero, and so that if you restrict to this space here, now it's a continuous function.

I will denote the ball by the same. But this here is actually a ball in a different metric space, because this is now a ball in the continuous functions on this smaller interval. But it's of the same radius. So if we restrict the map to this space here, then on this, it is a contracting map. So we need to cut down the integral.

So now, how do we do that? So we need the contracting property again. And so suppose you take two functions. These are continuous maps on some interval, say, from right here.

Again, delta is now-- that's a delta we are looking for. And it would be smaller than delta zero. So since it's smaller than delta zero, we already know that T-- so we have two maps like this. They are both supposed to be in the ball, so this means that this thing here is less than M. And we know we have already proven, just because delta is less than delta is zero, that T of this function here has norm less than or equal to M.

But now, let's look at the difference. So T of y1 of x minus T of y2 of x. And we're interested in the norm here. And now x here is in this smaller interval.

So now let's just write down what the map is. So the map here is the integral. So it's a. Sorry, it's a plus the integral from zero to x of f of y1 s plus g of s, ds, minus a plus the integral from zero to x of f of y2 s plus g of s, ds.

And now you see that the a's cancel. The a's cancel. And the g of s, and those also cancel out. So you are left with that this thing here is the absolute value of the integral from zero to x of f of y1 of s minus f of y2 of s, ds.

Now, so last time, when we proved that the map T takes this ball into the ball, we defined-- so let me just define that again. We defined L1. And L1 was the maximum of, actually-- I don't remember whether it was called L1. Let me just call it L to not call it. So L here is the maximum of f prime of z, where z here has norm less or equal to M. So this is so f prime exists and is continuous. So on this interval here, it has a maximum. So it's a nice finite number.

And so now, we use the-- so by the mean value theorem-- the mean value theorem-- by the mean value theorem we have that f of z1 minus f of z2, as long as z1 and z2 are in the interval from minus M to M, then we have that this difference here is equal to the derivative in some intermediate point set to sigma-- to psi, I guess. Psi-- times the difference between the two. z1 minus z2. So we have this. This is the mean value theorem. And so we have that-- therefore, we have that the norm here of the difference is bounded by this here.

When we take the absolute value of this, we're getting the absolute value. So this here is equal to the absolute value of f prime sigma times z1 minus z2. This thing here, because these two are in this interval, this is bounded by capital L. So you're getting this. And so now, if you use this factor, and you use this on this-- so this thing here is less than or equal to the integral from zero to x. And I can take the absolute value in under the integration sign. So I get f of y1s minus f of y2s, ds. Right?

And so now I can continue this string. So I have what I'm interested in is T of y1 at x minus T of y2 of x. And I'm interested in the bound for the absolute value. And I have that this is less or equal to the integral from zero to x of f of y1s plus f of y2s, ds. And now I know that this integral here, x is supposed to lie in this interval here. So this means that the length of this interval from zero to-- is at most delta. So delta.

And then I can take the maximum of this. Or I can take this one. I can take the maximum of this, but you see that this string here is bounded by. So this is the maximum. It's the maximum. And if I use this inequality here, it's bounded by L, and then it's applied. This here you should think about as z1. And this here for each x, s, you can think about this as z1 one and z2. And so you're getting this thing here times y1s minus y2s. And here, s will now lie in the interval, in this interval here. Oops. Right.

And so now you see that this is just a constant. So this is equal to delta L, and then this maximum s, these are equal to delta y1 minus y2, like this. This is evaluated in s, but I don't write that. And this thing here, but this is just delta L. This maximum here is just the distance between this function and that function. That's how we define this distance.

And so you see that this right hand side now holds for all x. So this means that we have that the maximum of this thing here over x, where x is in the interval between zero and delta. So this means that we have that the distance between T of y1 and T of y2-- this thing here is equal to the maximum. This is how we define that distance after all.

And then it's of this thing here. Oh, sorry. x minus T of y2 in x. So we have that this and this inequality say that it's less than or equal to delta L, d, y1, comma y2. And so you see that now if you-- so L here was just a fixed constant.

If I now set delta, just to make sure that there's no problem, I'm setting delta equal to 1 over 2L. I don't really need the one, but I'll just do it like this so I don't have zero in the denominator. If I choose the delta to be equal to this, well, then you see that this thing here, the inner product of these two is less-- or the product of these two is less than 1/2. So this is less than or equal to-- I can just replace it by 1/2, like that.

And so you see this is exactly what it means to be a contracting mapping. In fact, it's a contracting mapping, which is pretty nice that the constant is not just strictly some constant strictly less than 1, but it's actually half. So it's a contracting mapping theorem in contracting map. And so now by the contracting mapping theorem, we have a fixed point.

So this solves the array. And it does more than that, because the contracting mapping theorem says that it has a unique fixed point. So this means that-- and if you have a solution to this, if you have a solution to this, then we saw last time that if it is a solution, then it is a fixed point.

So now we almost have existence and uniqueness. Let me explain what is missing to the existence and uniqueness. So for that, let me backtrack a little bit and talk about something about metric spaces, specifically about connected metric spaces.

So suppose you have a metric space, then we say that it is connected. If x here is the only subset, so the whole set itself is the only subset of x, that is, it is both open and closed.

Remember that if you take a metric space then the-- sorry. If x and-- sorry-- and the empty set are the only subset of x that are both open and closed. Remember that if you take a metric space, and if you take the whole set or the empty set, those are all-- they are both always closed and open. And so it's connected if those are the only two subsets that are at the same time open and closed.

And so now, we have that. We have the following. We have the following lemma. Theorem. Let's call it a theorem that if you take an interval here with the usual metric here-- so I here is an interval-- then the theorem is that this metric space here is connected. So let's try to prove that.

So I take now a subset here that is both open and closed. And we want to show that if A here is non-empty, then it implies that A is everything. So we're assuming that it's non-empty, and then we should try to prove that A is everything. So we let x zero be an element in A, but we're assuming it's non-empty, so we can let x zero be an element in A. And now I want to define--

So I have here-- I have here my interval. This is my interval I. I have here x zero, and I'm looking at the y in A, so that if you take everything so that-- so y is an A. And I'm going to look at y with y bigger or equal to x zero. And I'm going to look at the y so that this whole interval here from x zero to y is contained in A.

So define as subset here, B of I, so that it's the y. The subset B is the y in A with the property that it is that y here is bigger or equal to x zero, and you have-- and this interval here is contained in A.

And now I want to show that. So let's say for the sake of argument, let me assume that this integral is a closed interval, like this. It works equally well. The argument obvious modification work equally well if it's an open or half open interval.

So I take some y here, and I have-- so I have a y, and y is-- so it's a set so that this whole interval is contained in A. And what I want to show is, so I want to show that y is in A, and this is the first thing I want to show. One is that y is in A.

And then the second thing I want to show is that y is equal to B. You see, if I had proven both of these properties, then it would be that-- so then it would mean that everything here would be in A, including this last point. And so this means that we have this entire interval is in A-- in an I, sorry. And then likewise, we could argue on the other side.

So now, how do we prove this? So well, the first thing is that if we take-- so if we set-- sorry. Sorry about this. This here should have been-- This here should have been y zero. And this should have been the soup of y in A. Soup of these guys. And then this here, and we want to prove that this thing here is in A. And the second thing is we want to prove that this soup is actually equal to B. That's what we want to prove. So the soup of these, well, if y zero is soup of elements, then you have a sequence, so here you have y zero.

And this means that you have a sequence of elements, which are then in elements in the set B, this is soup of B. This is the same as soup of B. Sorry. Soup of B. I could write it like this. So I take this set, and these are the y so that this entire interval is contained in A. And then I take the soup of this set, and I want to prove that this soup is in A, and then I want to prove that, in fact, the soup is B. So that is a soup of such elements in capital B means that there exists a sequence of elements in B, so that y here, y zero, is the soup of B.

This here implies that there exists a sequence yn. And this yn are in A, and you have that x zero, yn is contained in A. And then you have that these guys converge to y zero. This is what you get from that it's a soup. These two things are that the elements are in-- that these elements, so this here is just that the yn on B. So now, what you have, so this here means-- so here I have x zero. Here I have y zero. And I have these elements yn. And in particular, they are in A. But A is closed.

Since A is closed, since yn is in A, and yn and A is closed, it's a closed subset, this means that the limits here-- the limit y zero, which is the limit, this here must also be in A. So we have proven the first property. You've proven that. So we've proven that y zero is an A.

But now you see that we have that for each of the yn, we have that this interval here lie in A. And they converge to this. So it means that if you take anything to the left of y zero, well then eventually, there's some yn to the right of that. And so this whole interval is in A. So you see that this-- it now means that not only is y zero in A, but y zero is really in the set B.

This here only used that A was closed. So now we need to use that A is open. So now, suppose we take a element. Suppose so now here we have x zero. We have here B, and we want to show that this y zero we defined to be the soup of this set capital B, we want to prove that this thing here is, in fact, y zero.

Suppose not. Suppose not. So suppose that you have here, strictly between these two, this here is y zero. Now, we already know that y zero is in A. This is what we proved here.

But since A here is open, there exists some tiny little interval. There exists some interval here around this y zero. So if you take y zero, and you subtract this epsilon, so there exists some epsilon greater than zero, so that if you're looking at this tiny little interval centered at y zero, then this here is contained in A. This is just because A was open.

But this means that now you have that-- so this here is you assume this is y zero. So you have that this thing here lie in the set capital A. And this I guarantee you that you can actually go a little bit past it, that even if you go a little bit past it, if you go a little bit past it here, then these are still in A. So this means that this could not be the soup. So therefore, this point here had to be B. It had to be the final point. So this proved these two properties. And so once you have these two properties, you have that the interval is connected, that the only non-empty subset of the interval that is both open and closed is itself.

Now, why is this related to what we want? So this is just so we have-- suppose you take two functions. So why is this related to the uniqueness for the array? So this is the following lemma.

Suppose you have an interval. So this is a interval in R. And suppose you have two functions, U1 and U2. They're functions on the interval, and they are both continuous. And then assume that for some-- assume that for some x zero in the interval U1 of x zero is equal to U2 of x zero, so they coincide in one point.

And assume you have a second condition. And the second condition is that if U1 of x is equal to U2 of x, then there exists a neighborhood, so epsilon, where this is allowed to depend on x, so that U1 is equal to U2 in not just at x, but actually, on this little interval x minus epsilon, x plus epsilon.

So you have these two properties. So you have two continuous functions on an interval. They agree at one point. And if they agree at any point in this interval, then they agree on a little neighborhood of that point. But the neighborhood may depend on the point.

And then the statement is-- so the claim is that U1 is actually equal to U2 everywhere in the interval where they're defined. That's the claim. And so now, you see that if you take-- if you're looking at the proof of this is that suppose you prove-- suppose you define a function, and this function here is U1 of x minus U2 of x. If you define it like this, U1 and U2 are continuous. So f here is continuous.

You also have that f of x zero, where this is U1 of x zero minus U2 of x zero, and these two functions coincided at x zero, so this is zero. You also have that if f of x-- so if f of x is equal to zero, well, right now, then you have by this second property here, this property here-- so if f is zero, this is just meaning that the two function coincided at that x. So this means that they exist, and they exist at delta-- epsilon, sorry-- epsilon greater than zero, such that f of y is equal to 0 in this little interval around x. So you have that.

And now I claim that-- so now the claim is that the zero set, so the set where the function f is zero, this is both open and closed. So the claim here is That the set capital A, this is the set of x in the interval where f of x is equal to zero, that this here is both open and closed. We already know that this set, this is followup here, that x zero is-- so since x zero is in this set, then we have-- and the interval is connected, and I is connected, if this set is both open and closed, then we have that then A is equal to the whole interval.

So this means that f is constant zero, which means that the two functions coincided everywhere. And so we just need to prove that f is both open and closed. But the zero set, if you take a continuous function, so if f here is continuous, and you have a sequence so that each element here in the sequence is zero, and this xn converges to some x, well, then by continuity, by continuity, f of x would also be zero. This is by continuity.

So this means that this set here is definitely closed. And the open thing-- well, the open is just exactly this property that we're assuming that if you have an element in it, each element is interior, which means that there is this little interval around it. So we have proven that it's both closed. We prove that first, and also open. And so we have that if you have two such functions defined both continuous, and defined on an interval, they coincide at one point. And if they coincide at one point, then they coincide in a little neighborhood. If you have two functions with this property, then they must be equal in the whole interval.

So now, how does this relate to the uniqueness that we wanted to prove? So we're starting again. So we have the-- so this is, again, the Picard-Lindelof theorem, and this is the uniqueness part. And so you're looking at, again, you're looking at solutions to this array. And we have that now, we had that inner-- we have that inner, so and then we let so-- and let y1 and y2 be solutions. And let's say that these two, they may be defined on different intervals. Of course, but you just check the intersection of the two intervals that they're defined. So suppose that they're both defined, say, on the same interval, like this.

And this interval here and I, of course, has to include the zero where we want this condition. Well, then what do we have? So we have an interval, and we have two continuous functions. They're actually better than continuous. They're differentiable. And even the derivative is continuous. But never mind that. We have that y1 of zero is equal to A, and that's equal to y2 of zero. So they coincide at one point.

And now, the Picard-Lindelof theorem said that because there were two solutions, if you restrict the integral to looking at this little interval around zero, Well, then there was a unique fixed point. So this means that in a small neighborhood of the origin, so two in a small neighborhood of the origin. And again, this is thanks to that in the contraction mapping theorem, you had a unique fixed point in a small neighborhood of the origin, y1 is equal to y2.

But actually, the Picard-Lindelof-- the argument would work at any other point. Suppose you take any other point. And you take a solution to the array. Well, if you take another point, then the value-- so if you take-- so take any other point. So here you have zero. And so now take any other point. Let's call it x zero. And if you assume that these two functions coincide at this point. So suppose that y1 of x zero is equal to y2 of x zero. If you now restrict this interval, make it sufficiently small, then the argument of Picard-Lindelof tells you that you had this unique fixed point. So you had a unique solution on this little interval.

But since they coincide at this point, it means that they must be the same in this little interval. So you see that now we have the condition for the lemma that says that so you have these two continuous functions, and they were equal at one point. And if they're equal at a point, then they're equal in a whole neighborhood. So these two functions, these two solutions are equal on the interval where they're both defined.

Now, so now you might-- and then I want to do a little bit of review at the end and at least say something about the final. But before we get to that, let me just make one more observation about the Picard-Lindelof. So you may say that in the proof of the Picard-Lindelof, we kind of had to restrict the integral. Is that really necessary? Why should we restrict? Why should we only be able to say that we had a solution in a very small interval? And so if you're looking at the example, suppose you're looking at the example where f-- so f of y is equal to y squared, g of x is equal to zero. And so you're looking at the array solutions here.

Why? So that y prime is equal to y squared, and let's say equal to a for x equal to x zero. So suppose you're looking at an array like this. So at this x zero, you want the value of the function to be-- so you want-- sorry-- y of x zero equal to-- maybe this is a little bit better notation. So I want a function so that y prime is equal to y squared. And I want y of x zero equal to a. Now, let's say that x zero and a are both positive.

So if you're looking at this array, then you can just-- this one here is easy to solve, because you have a function. You have a function y prime equal to y squared. And now let's try to find a solution in a neighborhood of x zero. But there it's supposed to be positive. So in a small neighborhood, it's positive. So I can divide by y squared. So I can looking at it so this thing here is equivalent to that y prime over y squared is equal to 1. And this thing here I can write as 1 over y prime times minus 1, this thing here. If you're looking at this function here by the chain rule, it's equal to that.

And so this means that now you can apply the fundamental theorem of calculus. And if you're integrating this function here and you're integrating it from its two values x zero to x, d of s-- sorry, this is the prime-- then what you get is that this thing here is equal to, well, on one hand, by the fundamental theorem of calculus, it's 1 over y of x minus-- and then there's a minus sign. Sorry, this here, because of this minus sign. So it's minus this plus 1 over y of x zero. And on the other hand, it's equal to this. And when you're integrating this on this interval here, so it's also equal to x zero x, and then 1ds. This thing here is just x minus x zero. x minus x zero.

So if you solve it, then let me just write the result. If you check what this gives you, this gives you that when you solve it like this, you're getting-- you're getting that y here is-- so you have this array here. And you're assuming that these things here are positive, and then the solutions are of the form y of x equal to a, over 1 plus ax zero minus x.

And so what you should observe here is that this thing here, as this is not defined, so this is note that y is not defined on all of R. But instead, as x here goes to x zero plus 1 over a, as x zero goes to this, this thing here goes to minus 1, so the denominator goes to zero, so the function blows up.

So this here shows you that it does, in a neighborhood of this point, it has a solution. But that neighborhood is not-- it's some relatively small neighborhood, and the function, you cannot extend it past s, past s. A certain value, you can't extend it past. In this case, you can't extend it past this point here, so to the left of this point to the-- sorry-- to the right of that point. So now, I want to say a little bit about the final. Yeah, sorry.

AUDIENCE: Why is it that you can't-- once you reach that point where it blows up, it never solves the differential equation past that point?

TOBIAS COLDING: Yeah, that's right. So the thing is, so right. So this here, this array is going to force it to-- once this here, once y become large, then y become really, really large. So it becomes large too quickly, so it becomes quickly. So it will give you some of array like this that you have some critical value where you couldn't extend it past that, because it will have a tangent that is going vertical like this.

And so you can't-- so this means that there's-- that's right. So you have a solution that exists on some interval, but it doesn't exist past this point. So you can't have a solution on some interval larger than this to the left. And there's probably a similar phenomenon on the other side.

And it's a kind of standard problem, standard issue, that for differential equations, you can get blowups like this. That is a particular simple example. Any other questions? OK. OK, great.