[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: Last time, we started looking at what's called the Riemann integrals. For the Riemann integrals, we have a function, f, that is defined on some interval from a to b into R. And this function is bounded. And then we have P here is a partition of this interval.

And so this means that we have a number of dividing points, where a is the first, and then we have x1, x0. That's a, x1, x2, and so on, up to, say, xn. And that is the final point, end point, of the interval. So this here is a partition. These are the dividing points.

Then we look at the length of each of these subintervals. So we have the subintervals from xi minus 1 to xi. And the length of one of these subintervals we denote by delta xi. And so this is xi minus xi minus 1.

And then we look at-- so here, if you have the function here, this is the function f. And here, we have the integral from a to b. And so this here is the first dividing point. This is the last dividing point. And then we have these other dividing points in between. And this is maybe like that.

Then we look at the function on one of these subintervals. So we look at-- we look at the function. And we restrict it to one of these subintervals.

And then remember the function is bounded. And so we're looking at it on one of these. And we defined capital Mi to be the sup of f over this subinterval. Likewise, we defined little mi to be the inf.

And this makes sense, the finite number, because the function is bounded. And then we look at the upper and the lower sum. So the upper sum. So the upper sum is that we have this given partition. And we look at the sum here of the Mi times delta xi. And we're summing. This is from 1 to n. So this is the upper sum of the function for this partition.

And likewise, we look at the lower sum. This is where we sum the infimum. And we weight them by the length of the intervals. So we look at mi and then delta xi and i equal to 1 to n. And this is the lower sum for this partition.

And then just using-- then just using that-- so of course, for these mi, like these here, it's the inf and sum on the same interval. So little mi is less than or equal to capital Mi. But this also means that these weights are the same. If this here is less than that, it means that the lower sum is less than or equal to the upper sum.

Now, the second thing we looked at is-- so again, we looked at-- so again, we have this partition P. And we looked at P1 as a subpartition-- subpartition or a refinement.

This is if P1 has the same dividing points as P and then presumably some-- then-- I mean, then some more.

But I mean, you could have that-- you could have that the true partition was exactly the same. And you would still say that one is a trivial subpartition of the other. But in general, a subpartition would have more dividing points.

And so then the thing is that if you're looking at-- so let's say that-- so that P here, again, has the partitions. This is the partition that comes from P. And let's say that P1 here-- let me illustrate it as if P1 had one additional.

So here, you have x0, which is a. Here, you have xn, which is b. And then you have two other. Say this is x1 maybe. And this here is xk. Then you have a bunch of other. This is the partition p.

And then p1, let's say, have one more dividing point. It could have many more dividing points. But if you're looking at-- so you have here xi minus 1. And then you have xi and then maybe the P1, this partition, have an additional dividing point, which is called that one.

Well, then the contribution that you get from-- to the lower and upper sum-- so the contribution that you get from this or that for P, you can think about that as mi. This is the contribution from the lowest sum. And here, you have the contribution from the upper sum.

But now you see that for the subpartition, then you have that-- so the intervals for the subpartition-- this here is the interval for the original partition P. But for the subpartition, this is now divided into two intervals, this here and that here. And so you have that.

So this thing here, this is, in this case, split into two. And one of them is y minus xi minus 1. And the other is xi minus 1.

If you're looking at the-- say, the lower sum for this subpartition, so this thing here, so the contribution that you get from this here, where you have-- now you have two intervals. And so you're getting a-- you're getting a contribution, which is the inf on this thing here, on this little interval here of f, times y minus xi minus 1.

And then you get a second contribution, which is inf on this interval here, from y to xi of f. And then it's xi minus y. So this is the contribution that you get from this interval here.

But this thing here is smaller or equal to the-- it's actually-- sorry. This here is larger or equal to the inf where you take the inf over the larger interval that goes from this point to that point. So you have that this is larger than y minus xi minus 1. This thing here is also larger than this little mi. And then it's xi minus y.

And now you can just factor out-- so you're getting mi. And then when you add these two together, then you see that this gives you exactly xi minus xi minus 1. So this is little mi times delta xi. So this means that if you-- so you see that the contribution that you get on this little interval for this is larger than the contribution that you get from this-- larger, equal to.

And so when you're summing over all intervals, then you get that if P1 is a refinement or subpartition, then this lower here, the lower sum, go up. So this lower sum go up.

And if you do it now, if you do the same stuff for the upper sum, then this here would-- instead of being inf, it would have to be the sup. This here would have to be the sup. But the sup on the smaller interval is now then the-- this inequality would reverse. It's less than or equal to. And then it would come to sup on the larger interval. This here remains the same. This here again becomes a larger one.

So this here becomes less than this. This here becomes less than this. And so what you see is that the sup here go down. So the lower sum go up. And if you're looking at the upper sum, this here, decrease. So you have that.

And so putting this together, you have that the lower sum for the partition P is less than the lower sum for this subpartition, which is less than the upper sum for the subpartition, which is less than the original upper sum. So you have that.

And so this is like a key fact. It's an obvious fact, but it's a key fact. So that was the upper and lower sum. So now we define the upper and lower integrals.

So we define the upper and lower Riemann integrals. You define them as follows, that the upper one we denote like this. And this here is now the inf of all partitions of the upper sum.

And likewise, we define the lower Riemann integral of this function, this bounded function, to be the sup of the lower sums, like that. So sup of all partitions. So that is the upper and lower sums-- Riemann integrals. And so now we say that the function, this bounded function f, is Riemann integrable.

So f here-- so f is said to be a Riemann integrable. And we write that f here is like this. So it means that it's Riemann integrable.

If the following is the case, so if this lower integral here is equal to the upper integral, and if this is the case, then we just write-- and then the Riemann integral of f is then denoted by-- just like that.

And so this is so this is Riemann's way of defining integrals. There are other ways, but this is-- I think this was the first way where you have a definition that really made sense.

So now the first thing I want to observe is that for a general function, you always have-- so in general, so we always have-- whether the function is Riemann integrable or not, we always have that the lower integral here, the lower Riemann integral is less than or equal to the upper.

And so why is that the case? So this is because-- why? This is because if you take a partition-- so fix a partition.

Then you have that the lower sum here is, of course, less than or equal to this thing here, because this here was the sup of all partitions. So you have that.

But in fact, what you also have, that you-- but more importantly, you have that this L of f here, I claim that this is a less than or equal to the upper sum for any other partition, for any partition.

So I claim that you always have this here. If you take two partition P and P star, then the lower sum for one is less than or equal to the upper sum for the other. Why is that the case? This just uses that inequality over there. It uses that together with this inequality.

So let's see. So if you take-- if P here is a partition and P star is a partition, then you could take a partition-- so let P bar be the partition that has all the dividing points of P and all the dividing points of P star.

So the dividing point of P bar is the union of the dividing points of the two partitions. But this means that P bar here-- so P bar here is a refinement. It's a refinement of subpartition of both P and P star, because it has all of the dividing points of either.

But this means that-- remember that the lower sum-- so if you're looking at this-- the lower sum, if you take a subpartition, that goes up. So if you take a subpartition, the lower sum will go up. But the lower sum is always less than or equal to the upper sum for a given partition. But this here is now a refinement of P star.

So the upper sum goes down. So this is less than or equal to U f of P star. And so there you have it, that this thing here, the lower sum for one partition, is always smaller than the upper sum for the other.

And so this means that if you now take a partition-- so now it means that for a fixed partition P-- so if we have that this thing here is less than or equal to this thing about P as being fixed, but this is now for all other-- for all partitions P star. So this is for all P star.

But if it holds for all, this is-- sorry-- is less than or equal to. If it holds for all, then it-- also, this inequality, this is a fixed number. This means that this here is a lower bound for this family. So this means that L f of P is less than or equal to the inf of those guys. It's less than or equal to the inf.

But that here is just the-- this here is by definition the upper Riemann sum-- or, rather, Riemann integral. But this is for each. So now you forget about the P star. You have that this thing here is less than that number. So this means that this number here is an upper bound for this whole family.

So this means that also the sup, the smallest upper bound, must be less than that. So this means that the sup of L of f, p must be less than or equal to this thing here.

But the sup is by definition the lower integral. So you always have-- and this claim over here. So that's the first. So let's look at a function that is--

I added just a little bit, if you find it interesting. I find it interesting, but maybe you don't, but a little bit about Riemann, that he died rather young of tuberculosis. But he had an incredible impact. He had an incredible-- I mean, this is one of his contributions.

His first contribution is he was a student of another really famous mathematician, perhaps the most famous of all time-- Gauss. And so at those times, when you wrote your thesis, then you had like six months to write your thesis.

Formally, you couldn't choose your subject, but it was always-- anyway, you always got permission to choose it. You had to give three suggestions for something you would write your thesis on. But it was always-- it was a unspoken rule that your advisor always picked number one. And that's what you wanted. And then you could add some other crazy stuff as two other suggestions.

So Riemann had three suggestions. And the third one, which he had absolutely no idea about, was to make sense of curvature in higher dimension. And so he just put it in there. He knew it wasn't going to be selected.

But actually, at the time, this was what Gauss was interested in. And so Gauss said, to hell with the rules. I'll pick number three. And so he had six months, and nobody had figured this out. So Gauss is considered by many the greatest mathematician of all time.

But after six months, actually, so Riemann came up with something called the curvature tensor, which is this objects in all dimensions. And that generalized Gauss's work. And this lay the foundation for general relativity that came many years later, 70 years or so later.

So that was the Riemann's first. This is one of Riemann's other contributions. And there are other things that was only published a long time after his death that have had a tremendous contribution in other areas. Anyway, so this is-- so this was Riemann's way of defining the integrals.

Let me look at a function. So we talk about that a little bit. But we didn't actually define it. So here's an example of a function that is not Riemann integrable.

So it's a function defined on the interval from 0 to 1, so it's bounded. And so the function is f of x is equal to 1 if x here is rational. And it's 0 otherwise.

And so of course, this function here is always between 0 and 1. So it's definitely bounded. But if you take now any interval-- so this here, so I claim here f is not Riemann integrable.

And so the thing is that if you take-- if you take a interval like this and you take a subpartition like that, and now you have that-- and now you're looking at one of these mi. So here, you have xi minus 1. And here you have xi.

And so you're looking at mi. So this is some little interval like this. But on any kind of small interval, if it's really a proper integral, if it's not just a point but it's a proper integral-- interval, then there's always points that are rational and always points that are not rational.

So this means that this here, this here, the sup here, would have to be 1, but the inf here would be 0. And so this means that for the-- so this is just any partition. And so this means that for-- if you're looking at the upper sum here, but it's now the sum of these capital Mi delta xi. This here is 1. So it's just the sum of the length of the small interval. But that's just one because the whole interval is from 0 to 1.

And likewise, for the lower interval, lower sum, this is the sum here of this small mi delta xi. These here are 0. So this whole thing here is 0. So you see that if you-- this is for all partitions. So this means that if you take the infimum of this, you still get 1. And if you take the sup of these still get 0.

So those two numbers are not the same. The lower and the upper Riemann integrals are not the same. So the function is not Riemann integrable.

So now the next thing is-- the next thing is-- so we want to know which functions are Riemann integrable. And so here's a useful little criteria. So lemma-- so you have a function again that is bounded.

Then f is Riemann integrable. And again, we write it like this. So it's Riemann integrable if for all epsilon greater than 0, there exists a partition P so that the upper integral-- upper sum-- sorry-- so that the upper sum minus the lower sum-- remember that this is always bigger than the lower sum if this here is less than epsilon.

So if we want to check that a function is Riemann integrable, then we just need to check that for all epsilon, we can find a partition with this property. And so this is a statement that is if and only if-- so I should say if and only if-- if and only if.

So just try to prove that. And then we're going to use this later on to prove that a lot of functions are Riemann integrable. So suppose that f-- so if f here is Riemann integrable-- so if it is Riemann integrable, then we know that this lower Riemann integral is equal to the upper Riemann integral.

This here was the sup of all-- this here was the sup of all lower sums for all-- over all partitions. And this here was the inf of all upper sum, like that, over all partitions.

So therefore, given epsilon greater than 0, there exists a partition P1. Why? Because this here has a sup over all partitions.

So it means that there exists a partition so that this lower sum here is bigger or equal to this here minus epsilon over 2. It's just because it's a sup of all these numbers. So there must exist one that comes this close to the sup.

Likewise, there exists a partition P2-- sorry-- this here should be P1. This is the sup over-- sorry. So there exists a partition P2 so that the-- so that for that, this is one where this value here is really small.

So this is not too much larger than this. So this means that this for P2 is smaller or equal to this here, upper Riemann integral plus epsilon over 2.

So now let P be the partition with all the dividing points of both P1 and P2. So it has all of the dividing points from P1 together with all of the dividing points from P2.

But now for this partition, we have that-- well, for-- this is a refinement of P1. So if it's refinement, the value here goes up. And this thing here is-- so this is less than the upper for the same thing.

And since it's a refinement also of P2, we have that it is less than the upper sum for P2. But now you see that this thing here-- so L-- so the difference between these two-- the difference between these two numbers-- so this is the difference here. So U f P minus U f P-- sorry-- L f P, this difference here, this is bigger than that.

So if you take this and subtract this, you're getting something non-negative. And this, the spread between these two, is smaller than the spread between these. So this is less than U of f P2 minus L of f P1.

But this thing here is-- this thing here is less than U f P2 minus this upper integral here. Assuming that it's Riemann integral, so the upper and lower are the same. So this thing here we can write it as this plus minus L f P1 minus the integral here, like that.

I'm just subtracting and then adding back the same thing. And this one here, we already found that this here was bounded by epsilon over 2. This here is-- remember, we're assuming that it's Riemann integrable. So this here is the same thing as when you have the upper one.

And this thing here is-- this here is also bounded by-- this is actually right. This is negative, but it's-- so this thing here is bigger or equal to-- maybe I should write it as-- well, let me write it like this-- plus. And then I can write it as the integral from a to b of f of x dx minus L f of P1.

But this thing here was bounded by epsilon over 2 also. So epsilon over 2 plus epsilon over 2. So this is equal to epsilon. So again, if you want to test that a function is Riemann integral, you just need to find-- for a given, for any given epsilon, you need to be able to find a partition so that the upper sum for that partition is minus the lower sum for the same partition is less than epsilon.

And so now this leads us to the next theorem. So the next theorem is that any continuous function is Riemann integrable. So any Riemann sum-- so any continuous function-- any continuous function on an integral like this-- this is-- a and b are real numbers-- it is Riemann integrable.

So this is the next theorem that we want to prove. But in order to prove that, we need a tool. And so this is what's called uniformly continuous. So just to show this, we need the notion of being uniformly continuous. So what is that?

So in the proof, we need the notion of uniformly continuous. So that's the following.

And you have a function defined on some interval, say. It could be all of R, for that matter. So you have a function like this.

And f is said to be-- is said to be uniformly continuous if for all epsilon greater than 0, there exists delta greater than 0 such that as long as x and y are less than delta, then their images is less than epsilon.

So what is the difference between this and continuity? For continuity, you're always-- you're looking at-- a function's continuous if it's continuous at all points. And so it's continuous at a point. To be continuous at a point, it's like you're fixing one of these two. So it's like x is fixed. And then you're looking at if y is sufficiently close, then the value-- the difference between the value at x and y is smaller than epsilon.

But here, it's like that the same delta works for all x. You don't have to fix one of the two points. The same delta work. So let's see where this is not the case. So let me look at two examples.

So just to understand the difference-- so to understand the difference-- the difference between continuity and uniform continuity. Uniform, again, refer to that x is not fixed, or x0. Usually, we call it x0 and x.

So one of the two points was fixed. And uniform means that you don't need to fix. It's like the same delta works for all x0's.

And so here is an example that suppose you take the function f of x equal to x squared. This is now on R. Well, we already know that this function is continuous.

And so if it was uniformly so, if f was uniformly continuous, then for a given epsilon greater than 0, there would exist a delta so that the value at x and the value where you, say, added delta, this difference here would have to be less than if you want-- if you don't like it, you could say delta over 2 if you want it to be strictly the difference between. But that doesn't really matter. I'll just say like that.

This here would have to be the case. And this here would have to be the case for all x. But what is this here? This here is x plus delta squared. That's the value of the function in this. And this is the value of the function in x.

But this thing here, it's a squaring out. It gives you x squared plus delta squared plus 2x delta. And this thing here would have to be less than epsilon. Delta is positive. Sorry. This here, minus x squared, would have to be less than delta epsilon.

And so you get rid of these two. Delta is positive. So if you have this, you would definitely also have that so this would imply that-- or delta-- this here, delta square, I guess, is positive regardless. But you would have that 2x delta here, would have to be less than epsilon.

Now I'm using the delta is positive. So this here would have to be the case for all x. But if you just let x be delta is now supposed to be-- you're supposed to find some delta so that this here works for all x. So this means that delta is fixed.

But if now delta x becomes incredibly large, then of course, you don't have this inequality anymore. So this function is not uniform continuous. Yeah?

AUDIENCE: [INAUDIBLE]

TOBIAS COLDING: Because it only helps us. So if this here is less than that, this is positive. This means that this here surely is also less than epsilon. So that's one example.

The other example is-- and actually, let me draw a picture to just illustrate it. So if you take. So again the function roughly looks like this. This is f of x equal to x squared.

This meaning of that it's uniform continuous means that if you take some epsilon for any given epsilon, so if you're looking at a interval over on the y-axis here, so this is between-- this here is like a epsilon interval. And you're looking at-- so if you're looking at some point here, this here is this point here-- is taken to this here.

And now you want to look at some small interval. So that you end up in that interval. And the thing is that as you're moving further and further out this-- the length of that interval would have to be smaller and smaller. But you can't use the same uniform size. So that's why it's not uniformly continuous.

The other example I want to look at is where you take a function-- where you're taking f of x equal to 1 over x, and x here is now in the interval, from 0 to 1. 1 is included, but 0 is not. And so there, the picture is that this here is the function.

And so you see that as you're getting closer, here is 1. And as you're looking at things closer to 0, then even a really small interval, the functions, the value of the function would deviate a lot on even a very small interval. So you couldn't use the same interval size to say that the value of the function is smaller than some epsilon. So just make that precise.

If you take a sequence xn equal to 1 over n and yn equal to 1 over 2n, then-- and you're looking at-- you're looking at the difference, so if you're looking at xn minus 1, here, you have-- we don't even need to calculate anything.

For the first thing here, you have xn. Here, you have yn. So this means that the difference between these two is certainly bounded by this here. So of course, you always have-- you have actually something much better. But this is actually smaller than 1 over n.

And if you're now looking at the value of the function, this thing here is 1 over xn. So that's n. And f of yn is 1 over 2n. So this is 2n. So this means that f of xn minus f of yn here, this here is equal to this difference here, is equal to 2n.

And so you see that you couldn't-- it's not like-- it's not-- for any given epsilon, it doesn't work that if these two points are, doesn't exist a delta so that if these two points are delta closed, then the images is at most epsilon. Yeah?

AUDIENCE: I don't know if this is too broad of a generalization, but is that essentially just the same thing as saying that the slope is bounded.

TOBIAS COLDING: Yeah, it's like that. I mean, it's sort of like that. I mean, it's not-- I mean, uniform continuity doesn't require-- I mean, sort of like that but not exactly, because uniformly-- slope is something about being differentiable, and you can have a uniformly continuous function that is not differentiable. But it's definitely related. Yeah.

So anyway, so uniform again means that for a given delta-- for a given epsilon, you can use the same delta for-- at all points. Otherwise, for continuity, you are always fixing one of the points. And then you were looking at other points close to that point.

So now the first thing that we want to prove is-- so we want to prove that-- so we want-- our goal here is to prove this theorem. Let's say that if you have a continuous function on a closed interval-- so a closed bounded interval, compact interval, from a to b, then the function is actually Riemann integrable.

And to prove that, we need again this notion of uniformly continuous. And so the first thing we want to prove is we want to upgrade continuity to uniform continuity. So what does that mean?

So it means that if you take a function on a compact interval like this-- so if you have a continuous function, then f is uniformly continuous.

So if the interval actually is compact, then the-- and then a continuous function is uniformly continuous. Let's just look at the two examples we had. We had two examples. In the first one, f of x was equal to x squared. But it was defined on all of-- it was continuous of course. But the thing is it wasn't on a compact interval. It was on all of R.

In the second, this function here is also continuous, but it's not a compact interval, because 0 is not included. Yeah?

AUDIENCE: Compact in the sense of the before?

TOBIAS COLDING: So remember that-- so compact, you can either think about it as-- we talked about compact metric spaces or compact sets. But for a subset of the real line, compact is the same as saying that it's bounded and closed.

And so this thing here, when I'm writing it like this, a and b are real numbers. And they are both included. And again, if it was this here, then one end point is included, but the other one is not. So it's not actually compact.

And so if the function here is continuous, then it's uniformly continuous. So that's what we want to prove. Again, if the interval is compact, then actually, you get the stronger continuity.

So now let's try to prove that. Suppose not. If this here was not the case, then it meant that-- so we take the negation of the definition of uniform continuity. So this means that there exists an epsilon. So suppose not.

Then there exists epsilon greater than 0 such that no delta would work, such that for all delta greater than 0, there exists x comma y so that the difference between these two is less than delta and the images is bigger or equal to epsilon.

That's just the negation of the statement. But now we choose a sequence of delta. And so we do this-- so this is for all delta. So we should take advantage of that. We look at a sequence of such delta. So we look at a sequence of delta. And this is where we use this for delta equal to 1 over n. So n here is, again, a positive natural number, like that.

So this means that we-- for each n, you have xn, and you have yn. They're both in the interval from a to b. And you have that xn minus yn is less than 1 over n. And you have that the images f of xn minus f of yn is bigger or equal to epsilon. Epsilon was positive, strictly positive.

But this means-- so now we have xn. So these are in this interval. So this means that we can find a subsequence by compactness. We can find of the Bolzano-Weierstrass theorem. We have a subsequence that converges to some x. This is just a subsequence of these xn's.

But now you can just look at-- so now, if you're looking at the corresponding subsequence of the yn's, so you have that yn k minus x, this thing here is less than or equal to-- by the triangle inequality, this is less than or equal to yn k minus xn k plus xn k minus x.

And we have that this thing here, this here, was less than 1 over n. And we have plus this here. But this thing here is actually over nk. Sorry. nk.

But you have that this thing here, this here, this subsequence go to x. So this here goes to 0. And this here goes to 0. So this means that these yn k actually also converge to x.

But now you have-- so you have this, and you have that yn k converge to x. So now this here together with that the function is continuous means that f xn k converges to f of x. And you have that f yn k converge to also f of x. This is just because since f is continuous.

But now you just look at-- now you just look at the-- so now you have that-- so again, f xn k converges to f of x. And this here converges to f of x.

But I claim that this here-- but we also have that f xn k minus f yn k-- we have that this thing here is bigger or equal to epsilon. But then you can't have-- if this here-- this here is clearly a contradiction, because if this converges to that, this here converges to the same thing. Then these two things here must get closer and closer together. But they're saying that they are some fixed thing apart. So that's clearly a contradiction. So this proof that if it is continuous, it's also uniformly continuous.

So now let's use this to prove that if you take a function-- so now we are ready to get back to the theorem that was our main interest. So you see also in the two examples I had, it was no accident, again, that in the first one, the interval was unbounded. In the second one, the interval was bounded, but it was not closed. And so either case, it failed to be compact. And that was why you could have a continuous function that wasn't uniformly continuous.

And so now we get back to this theorem. And so the theorem again is-- the theorem gives you this nice criterion for when a function is Riemann integrable. So if f here is continuous, then f here is Riemann integrable.

And again, this here should be-- it's understood that it's f here is defined on an integral like this integral. And so now the proof is like this.

We showed earlier that we only-- that we need only show that a given epsilon greater than 0, there exists a partition so that if-- such that the upper sum minus the lower sum for this partition is less than epsilon, that's all we need.

We need to prove that for each epsilon, there exists a partition so that the upper and the lower sum are within epsilon of each other. So now using that-- so now-- so given epsilon greater than 0-- since f is uniformly continuous-- so actually, we're proving that f is continuous on this compact interval. It's also uniformly continuous.

Since f is uniformly continuous, it means that there exists delta greater than 0 such that if x minus y is less than delta, then-- let's say delta is, let's say, less than 2 delta.

Then the images-- then f of x minus f of y is now smaller than-- and then instead of epsilon, I'm using epsilon divided by the length of the interval. Sorry. And this would be strictly less.

So now we let P here be a partition where each delta xi, each little interval here, is equal to delta. You just have to be, say, less than or equal to delta, but couldn't make it equal to delta.

So now if this is the case-- but then we have-- so now if we're looking at one of these small intervals here-- so now, if you're looking at one of these small intervals here, here, you have xi minus 1. And here, you have xi. So if you're now looking at one of these small interval, then you have that the function is continuous here.

So if you're looking at the sup of this interval-- so the sup on this interval, this is the sup of one of these intervals here, of f. The sup here, this is-- it's actually achieved by the extreme value theorem. This is achieved at some point if, say, ya-- y and maybe this-- my-- let's just say y.

And likewise, on this little interval, the inf here is also achieved. And let's call that y star. So the infimum is also achieved. And that's achieved at some y star.

But now this interval here have length. This here is at most delta. So this means that the difference in the value of any such two points so they are actually less-- strictly less than 2 delta. So that means that the value differ at most like this.

So it means that f of y, which is actually the sup-- so this here-- so since it's a sup, I don't even need this here. It's larger than that. So I don't need to even take absolute value sign on it. But it means that the difference here is now less than epsilon over b minus a. And so this here again is M of i minus m of i, like that.

And so now you see that-- so now let's look at the contribution. So for the upper sum, we have that this here is this capital Mi and then delta xi. For the lower sum, we have that this is the sum of the small mi delta xi.

And so now let's look at the difference between these two. That's the one we need to prove is strictly less than epsilon. So if you're looking at this U f of P minus L f P, this here is now the sum here. And then it's-- so you're summing-- this thing is the same. So you can write the sum here-- the difference. Well, let's just write it first. Just write it out like this.

But then now I can-- once I have written it out like this, see that I can write this thing here is the sum of capital Mi minus little mi times delta xi. And this thing here I already figured-- I already have a bound. So this is each of these is less than this. So I just write in that bound.

So this here is less than-- and then the sum here, epsilon over b minus a-- and then like this. But now this is just a constant. So this just-- I can just take this out of the summation sign. And so I get this thing. So this is the sum of the small interval. But the sum of the small interval is the whole interval.

So this is epsilon b minus a times b minus a. And so this here is epsilon. And so you have exactly-- it's a-- this difference here is strictly bounded by epsilon.

So let's prove that if the function is continuous, then it is on a compact interval. Then it is Riemann integral. And now let me at least state some of the basic formulas for Riemann integrals. And so maybe we might prove some of these next time.

So the first one is that if you take-- so you have this theorem here. The first one is-- so suppose you take a function that is Riemann integrable on some integral like this. And c here is constant. It's a real number.

Then this constant times f is also Riemann integrable. And the integral of this function here is equal to c times the integral, Riemann integral, like that. So that's the first one.

The next one is that if you take two functions, f and g. Suppose you take two functions, f and g, that are Riemann integral. And now let's look at the function that is the sum of these two. So this is a new function.

And the claim is that this here is also Riemann integrable. And the integral of the sum is the sum of the integrals. Oops. That's the second one.

And the third one is that-- so the third one is that if you take a function-- two functions, again, they are both Riemann integrable. And you have that f here is less than or equal to g. Then the Riemann integral of f is less than or equal to the Riemann integral of g, like that.

And the fourth rule is that-- the fourth rule is a little bit different. So this is if you split the integral. So the fourth rule is like this. Again, I will prove some of them, not all of them, but I'll prove some of them next time but not all of them. I won't prove the first one for sure.

So the fourth is that if you take-- so imagine-- so you have this integral here from a to b. But imagine that-- so here, you have a. Here, you have b. Imagine that c is somewhere in between a and b.

And suppose that you have a function that is Riemann integrable on this interval. Then the claim is that then, if you take the function and you restrict it to either of this interval or that interval, it's still Riemann integrable.

And f here is also in-- from c to b. And the claim is that the Riemann integral on the original integral is the sum of the Riemann integral, like that.

And so if you think about it in terms of area below the graph, if you take a function like this, if this here was-- if this here was c, then the area below the graph here, I guess this here would be sign area below the graph. But if you just took the area below the function was actually positive and you think about the Riemann integral as the area below the graph, if this here is c, this here is a, this here is b, then this area here plus this area is-- the sum of these two is the sum underneath the whole graph. So it makes sense.

Any questions? OK, great. See you next time.